MODERN CONTROL SYSTEMS NOTES

AKSHANSH CHAUDHARY

Modern Control Systems Notes, First Edition

Presented by: Akshansh Chaudhary
Graduate of BITS Pilani, Dubai Campus
Batch of 2011

Course content by: Dr. R. Mary Lourde
Then Faculty, BITS Pilani, Dubai Campus

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More content available at: www.Akshansh.weebly.com

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Introduction

* Pre-knowledge :-

* Laplace transform : $s = \sigma + j\omega$

  σ → amplitude; ω → freq.

  → Imp: assuming initial cond^{ns} = 0

* Fourier transform : $j\omega$ → only freq.

* z-transform : k → discrete sys

* In control sys: the analysis of STABILITY is done
  * Classical open loop sys. are inherently STABLE
  * +ve feedback sys. is generally UNSTABLE
* linearity of a sys. is defined by :
  ✔ homogeneity & superpos^{n}.
* time varying sys :- eg. in a missile, fuel burns, mass varies with time.

Imp * STATE SPACE SYS IS 1^{st} ORDER Differential SYS

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STATE VARIABLE

Imp remember the convention

* Denotion :- States variables represented by x.
  for n states :- $x_1, x_2, x_3, \ldots x_n$
  Inputs represented by u.
  Outputs represented by y.

time varying

In box representation
Arrow line → Thin : single i/p or o/p
→ Thick : multiple i/p or o/p

* state variables describe the future response of sys, given the present state, excitⁿ i/ps & the eqⁿˢ describing dynamics.

* Making order 2 ⟶ order 1

eq :-

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + kx = u(t)$$

let $x_1(t) = y(t)$

$x_2(t) = \frac{d}{dt}y(t)$

$$\Rightarrow m\frac{d}{dt}x_2 + cx_1(t) + kx = u(t)$$

* Defining State of a Sys :- (for n states)
  System matrix :- square $n \times n$
  i/p matrix :- not sq. $n \times m$

* No. of state variables should be min. & should be able to explain dynamics of sys.

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* State eqns :-

for any MIMO sys

dynamics of sys

i/p matrix

State D.E : $\dot{x} = Ax + Bu$.

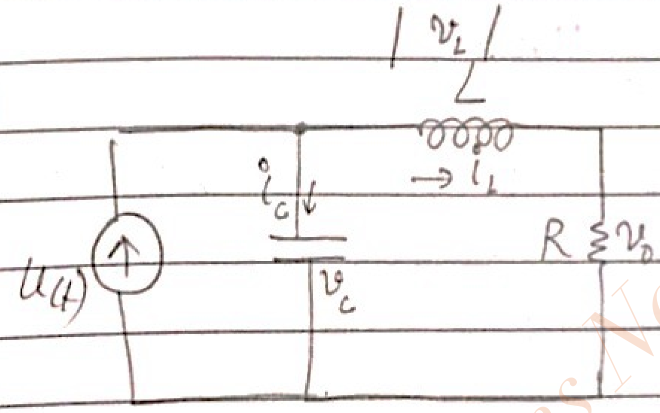
o/p eqn : $y = Cx + Du$.

feed forward component (usually zero → not feed forward)

→ Directly affecting o/p

*

(eg)



here,

$v_c = x_1$

$i_L = x_2$

o/p is at v_0. $\Rightarrow v_0 = y = (Cx + D)$

$y = v_0 = R\, i_{L(t)}$ (no feed forward path)

Objective :- Make matrix form (state space form)

$$y = C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow y = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

} O/p vector

$$(= R x_2 = R i_L)$$

$$i_c = C \frac{dv_c}{dt} = u(t) - i_L$$

$$L \frac{di_L}{dt} = -R i_L + v_c.$$

} Write A & B matrix from these eqns.

We have :- $\dot{x} = Ax + Bu$

State dynamics vector $\Rightarrow \begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/C \\ 0 \end{bmatrix}}_{B} u_{(t)}$

$$\frac{dv_c}{dt} = \frac{1}{C} u_{(t)} - \frac{1}{C} i_L$$

$$\Rightarrow \dot{v}_c = \left(-\frac{1}{C}\right) x_2 + \frac{1}{C} u_{(t)}$$

$$\& \quad \frac{di_L}{dt} = -\frac{R}{L} i_L + \frac{1}{L} V_c$$

$$\Rightarrow \dot{i}_L = -\frac{R}{L} x_2 + \frac{1}{L} x_1$$

Basically, given any control sys, we have to make the state space model :

Problem making TF from given vector eq^{ns} :-

$$\mathcal{L}(\dot{x}) = sX_{(s)} - x_0 = aX_{(s)} + bU_{(s)}$$

$$\Rightarrow X_{(s)} = \frac{x(0)}{s-a} + \frac{b}{s-a} U_{(s)} \quad \rightarrow (A)$$

$\mathcal{L}^{-1}(X_{(s)})$

$$= x(t) = e^{at} x(0) + \mathcal{L}^{-1}\left[\frac{b}{s-a} U_{(s)}\right]$$

A : represents sys. dynamics

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$$\Rightarrow x(t) = e^{at} x(0) + b \int_0^t e^{a(t-\tau)} u(\tau) d\tau .$$

Now, we want solⁿ in terms of matrix

Now,

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots + \frac{A^k t^k}{k!} + \dots$$

identity matrix

A matrix

$$\left(e^{x} = 1 + x + \frac{x^2}{2!} + \dots \right)$$

converting to matrix form.

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

(A) in matrix form :-

$$X(s) = \frac{x(0)}{sI - A} + \frac{B}{sI - A} U(s)$$

$$= [sI - A]^{-1} x(0) + [sI - A]^{-1} B U(s)$$

Let $[sI - A]^{-1} = \phi(s)$, s.t $\phi(t) = \mathcal{L}^{-1}(e^{At})$

describes natural response of sys.

So,

$$x(t) = \phi(t) x(0) + \int_0^t \phi(t-\tau) B u(\tau) d\tau$$

(initial condⁿ) (current state)

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Now $\dot{x} = Ax + BU$

$$\mathcal{L}(\dot{x}) \Rightarrow sX_{(s)} - x_{(0)} = AX_{(s)} + BU_{(s)}$$

$$\Rightarrow [sI - A]X_{(s)} = BU_{(s)}$$

$$\Rightarrow X_{(s)} = [sI - A]^{-1} BU_{(s)}$$

$$= \phi_{(s)} BU_{(s)}$$

& $y = Cx$

$$\mathcal{L}(y) \Rightarrow Y_{(s)} = CX_{(s)}$$

$$\Rightarrow Y_{(s)} = C[\phi_{(s)} BU_{(s)}]$$

Now,

$$\text{TF}, G(s) = \frac{Y_{(s)}}{U_{(s)}}$$

or $R(s)$: reference i/p

$$\Rightarrow \boxed{G(s) = C\phi_{(s)}B}$$

$$\hookrightarrow \phi_{(s)} = [sI - A]^{-1}$$

TF of a 2nd ord. RLC Circuit :

Solving further :

$$sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix}$$

Note :- $A^{-1} = \frac{1}{|A|}\,\text{adj}\,A$

adj A → Adjoint A

|A| → determinant

$$[sI - A] = \begin{bmatrix} s & 1/C \\ -1/L & s + R/L \end{bmatrix}$$

$$[sI - A]^{-1} = \phi(s) = \frac{1}{\Delta(s)} \begin{bmatrix} s + R/L & -1/C \\ 1/L & s \end{bmatrix}$$

A^T (A transpose)

$$\Delta(s) = s^2 + \frac{R}{L}s + \frac{1}{LC}$$

determinant of $[sI - A]$

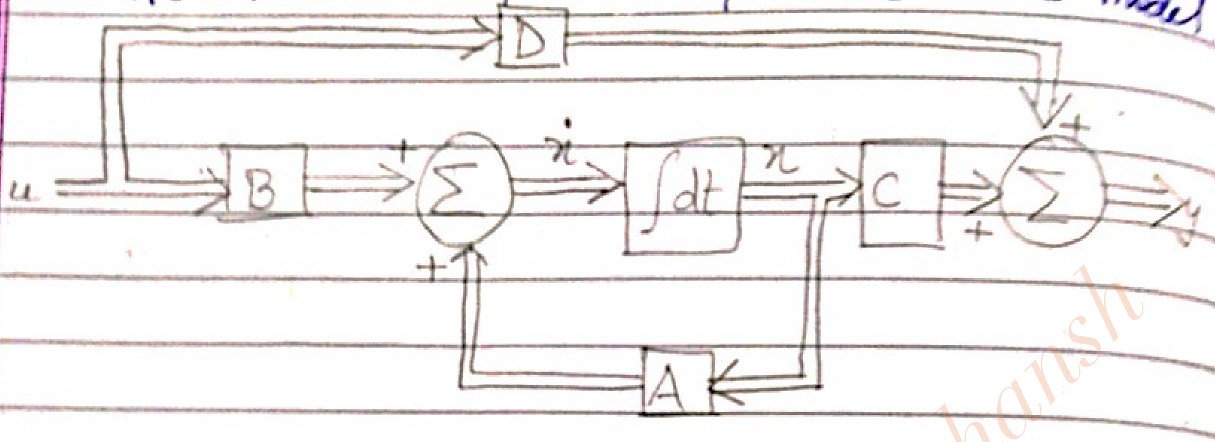
TF

$$G(s) = C\,\phi(s) \times B$$

$$\Rightarrow G(s) = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} \frac{s + R/L}{\Delta(s)} & \frac{-1}{C\Delta(s)} \\ \frac{1}{L\Delta(s)} & \frac{s}{\Delta(s)} \end{bmatrix} \begin{bmatrix} 1/C \\ 0 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{G(s)}} = \frac{R/LC}{\Delta(s)} = \frac{R/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

* BLOCK DIAGRAM Representⁿ of STATE SPACE model



* Given a D.E, how to derive state space model?

Consider a D.E :-

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_0 y = b_0 u \quad \text{(1)}$$

$\hookrightarrow$ y : o/p ; u : i/p

Choose

$$x_1 = y$$
$$x_2 = \frac{dy}{dt}$$
$$x_3 = \frac{d^2y}{dt^2}$$
$$\vdots$$
$$x_n = \frac{d^{n-1}y}{dt^{n-1}}$$

from (1)

$$\frac{d^n y}{dt^n} = b_0 u - \left[a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_0 y\right]$$

$$\Rightarrow \frac{d^n y}{dt^n} = b_0 u - \left[a_{n-1}x_n + \dots + a_1 x_2 + a_0 x_1\right]$$

Also,

$$x_2 = \dot{x}_1$$
$$x_3 = \dot{x}_2$$
$$\vdots$$
$$x_n = \dot{x}_{n-1}$$

$$* \quad \dot{x}_n = -a_0 x_1 - a_1 x_2 \dots - a_{n-1}x_n + b_0 u$$

Generalised form, $\vec{\dot{x}} = \vec{A}\vec{x} + \vec{B}\vec{u}$

$y = CX$

The matrix becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & & & \ddots & \\ & & \vdots & & & & \\ & & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Controlled o/p

Q. Given a TF :- $\dfrac{C(s)}{R(s)} = \dfrac{24}{s^3 + 9s^2 + 26s + 24}$

reference i/p

✓ Cross multiply & solve.

✓ Take inverse LT.

$\Rightarrow \dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$

Now, select state variables.

$$x_1 = c, \quad x_2 = \dot{c}, \quad x_3 = \ddot{c}$$

$$\dot{x}_1 = x_2$$

indicates -ve f/b sys.

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

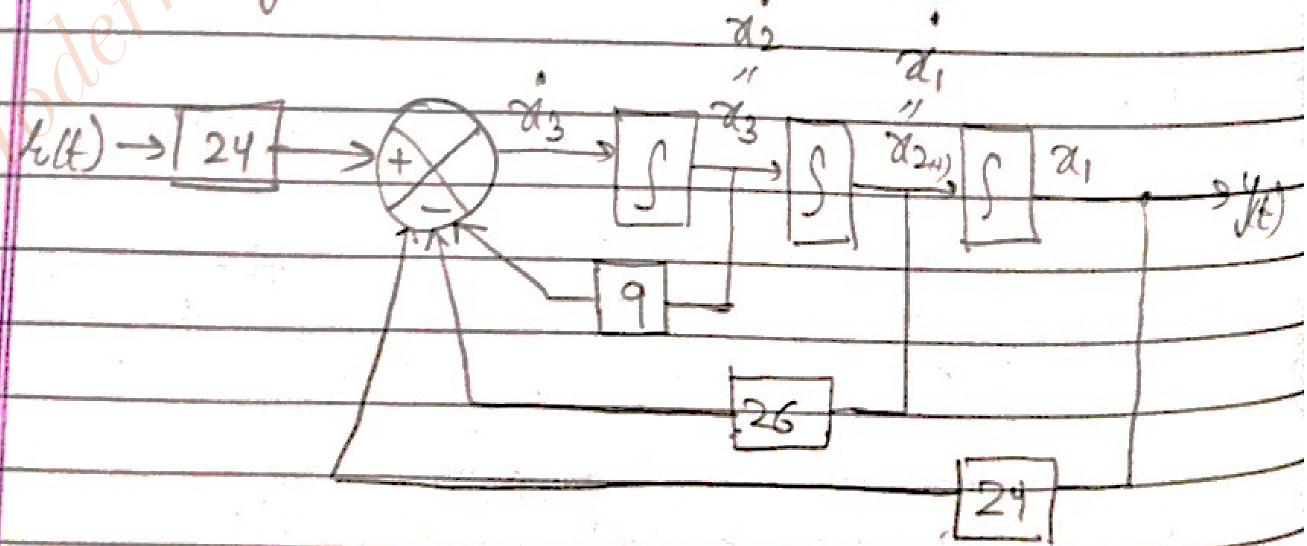
$$y = c = x_1$$

Forming matrix

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24r \end{bmatrix} [u]$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Block diagram



Decomposing a TF.

$$R(s) \longrightarrow \boxed{\frac{b_2 s^2 + b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}} \longrightarrow C(s)$$

$$R(s) \longrightarrow \boxed{\frac{1}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}} \xrightarrow{X(s)} \boxed{b_2 s^2 + b_1 s + b_0} \longrightarrow C(s)$$

Internal variables, $X_2(s)$, $X_3(s)$

o/p is represented in terms of state variables $X(s)$

Now, o/p, $Y(s) = C(s) = (b_2 s^2 + b_1 s + b_0) X_1(s)$

$$\downarrow$$

$$y(t) = b_2 \frac{d^2 x_1}{dt^2} + b_1 \frac{dx_1}{dt} + b_0 x_1$$

$$\downarrow$$

o/p in terms of State var. in time domain $\left\{ y(t) = b_0 x_1 + b_1 x_2 + b_2 x_3 \right.$

Q. Making block diagram from TF :-

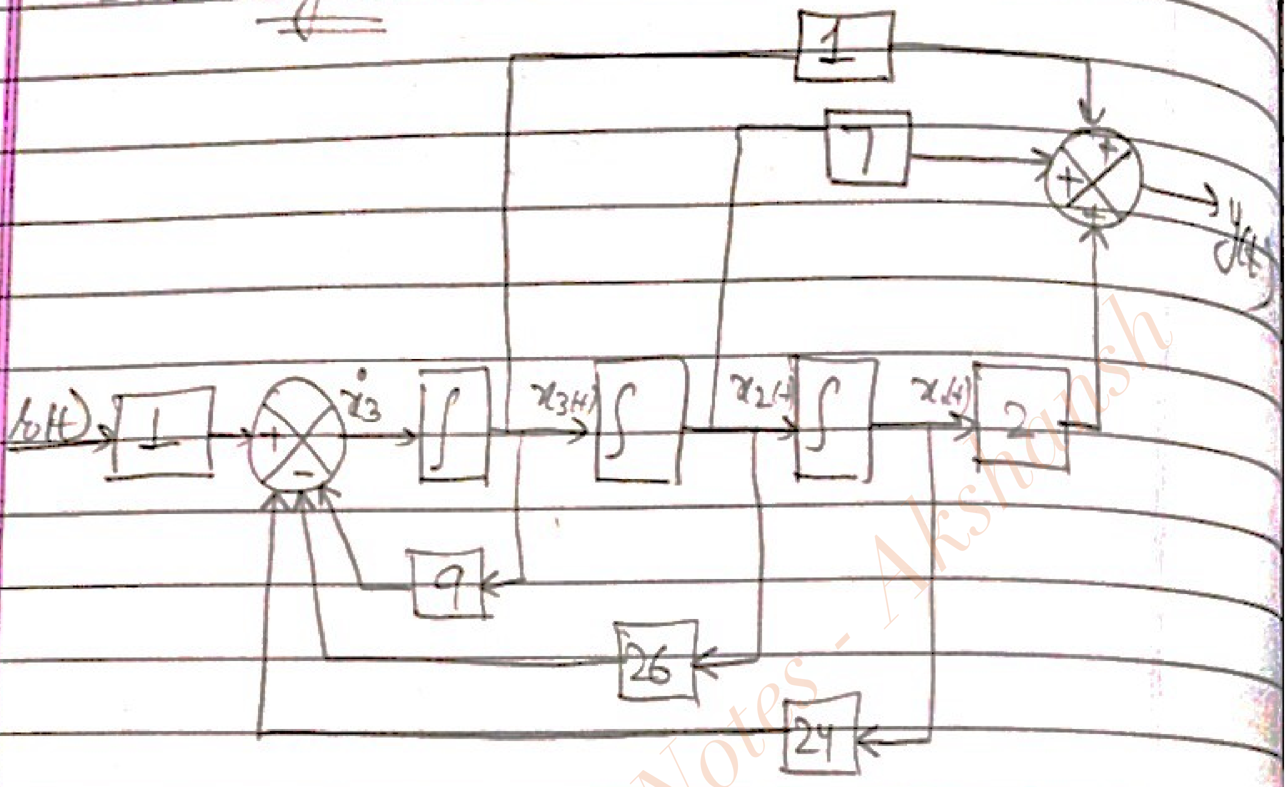
$$R(s) \longrightarrow \boxed{\frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}} \longrightarrow C(s)$$

$\Rightarrow$

$$R(s) \longrightarrow \boxed{\frac{1}{s^3 + 9s^2 + 26s + 24}} \xrightarrow{X(s)} \boxed{s^2 + 7s + 2} \longrightarrow C(s)$$

Int. var :- $X_2(s)$, $X_3(s)$

Block diagram:-



✯ STATE SPACE TO TF.

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Laplace transform

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$\rightarrow \quad X(s) = [sI - A]^{-1} BU(s)$$

$$\rightarrow \quad Y(s) = \left[C[sI-A]^{-1}B + D\right]U(s)$$

$$T(s) = \frac{Y(s)}{U(s)} = C[sI-A]^{-1}B + D$$

for f/b

eg
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \;\; ; \;\; D = 0$$

$\rightarrow$ f/b, $D = 0$
feed forward, $D \neq 0$

Now,
$$[sI - A] = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{|sI - A|} \left([sI - A] \right)^{\text{adjoint}}$$

$$\Rightarrow [sI - A]^{-1} = \frac{\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1}$$

Now, $TF = C[sI - A]^{-1} B$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{1\times 3} \begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix} \times \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$TF = \begin{bmatrix} s^2+3s+2 & s+3 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow TF = 10(s^2 + 3s + 2)$$

Note :- Finding Adjoint of a matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

Taking Transpose of this matrix

Q. Given a TF, how to find impulse response.

$$X(s) = \frac{1}{s(s^2+3s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}.$$

We know, TF, $G(s) = \frac{Y(s)}{R(s)}$

When $i/p = \delta(t)$, o/p = impulse response

$\Rightarrow$ when $i/p = \mathcal{L}(\delta(t))$, $o/p = \mathcal{L}$(impulse resp.)

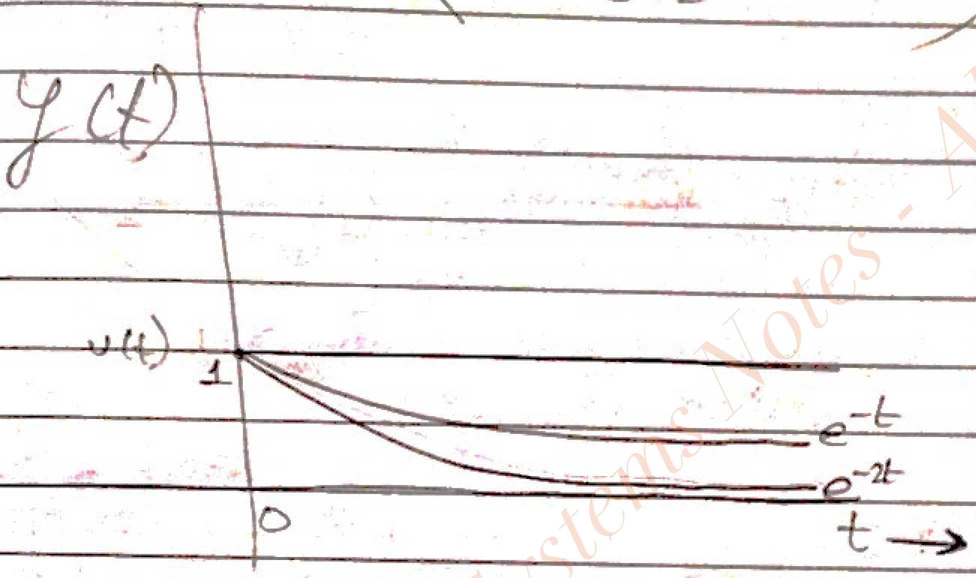
$\Rightarrow$ when $i/p = 1$, $o/p = \mathcal{L}$(impulse resp.)

$\Rightarrow G(s) = \frac{Y(s)}{1}$ $\Rightarrow Y(s) = G(s)$

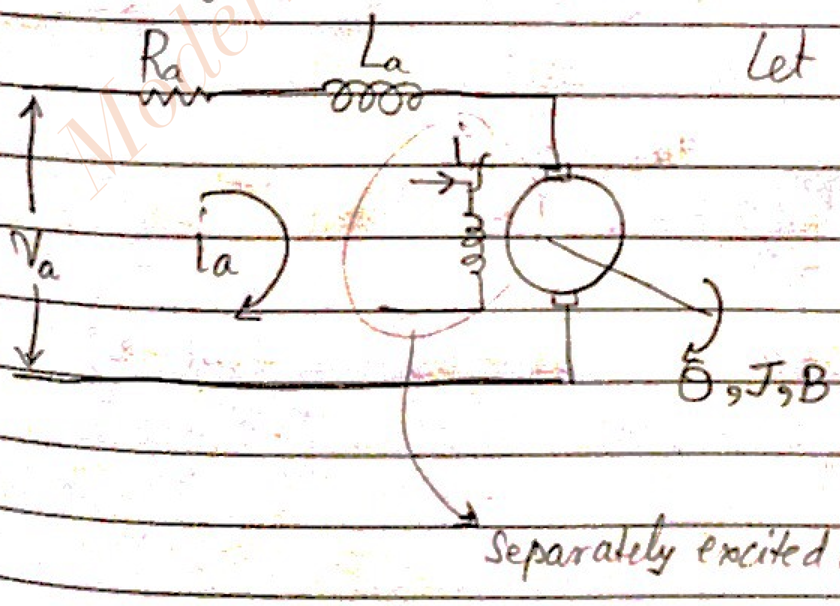
$\Rightarrow y(t) = \mathcal{L}^{-1}(G(s))$

So, $y(t) = A\,u(t) + Be^{-t} + Ce^{-2t}$

$$\left(\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s}\right) &= u(t) \\ \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) &= e^{-t} \\ \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) &= e^{-2t} \end{aligned} \right)$$



Q. Write DE & derive TF for given ckt. Also represent by block diagram.



Let $x_1 = \theta,\ x_2 = \dot{\theta},\ x_3 = i_a$

$\omega = \dot{\theta} \left(\frac{d\theta}{dt}\right)$

back emf

$$V_a = i_a R_a + L\frac{di_a}{dt} + E_b$$

Taking LT

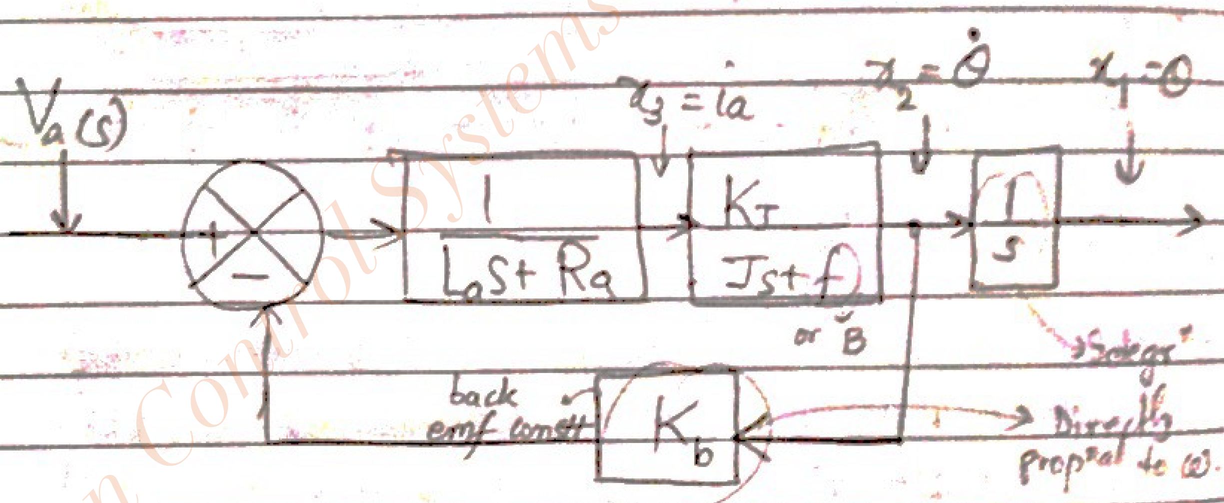
$$\Rightarrow V_a(s) = I_a(s) R_{a(s)} + L\left[s \cdot I_a(s) - i(0)\right] + E_b(s)$$

0

$$\Rightarrow V_a(s) = I_{a(s)} R_{a(s)} + sL_{(s)} I_{a(s)} + E_b(s)$$

$$\Rightarrow \frac{V_a(s) - E_b(s)}{I_a(s)} = R_{a(s)} + L_{(s)} s$$

$$\Rightarrow \frac{I_a(s)}{V_a(s) - E_b(s)} = \frac{1}{R_{a(s)} + L_{(s)} s}$$



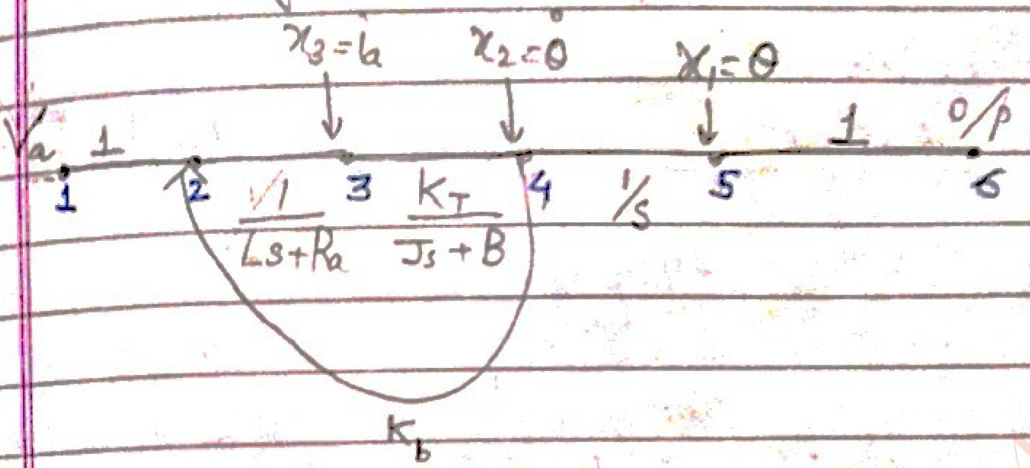
$$\text{Torque, } T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_T I_a$$

Taking LT

$$\Rightarrow (Js^2 + Bs)\theta = K_T I_a$$

$$\Rightarrow \frac{\theta}{I_a} = \frac{K_T}{Js^2 + Bs} = \frac{K_T}{s(Js + B)}$$

Making signal flow graph from block diagram



Writing state space represent^n :-

We have eq^{ns} :-

$\dot{x}_1 = x_2$

$J\dot{x}_2 + f x_2 = K_T x_3 \Rightarrow \dot{x}_2 = 0 \cdot x_1 - \frac{f}{J} x_2 + \frac{K_T}{J} x_3$

$V_a - K_b x_2 = R_a x_3 + L_a \dot{x}_3 \Rightarrow \dot{x}_3 = 0 \cdot x_1 - \frac{K_b}{L_a} x_2 - \frac{R_a}{L_a} x_3 + (V_a / L_a)$

Using them,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -f/J & K_T/J \\ 0 & -K_b/L_a & -R_a/L_a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} V_a$$

O/P, $y = \theta = x_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Date 17/9/13

→ Prev. knowledge req^d :- DC m/c working & formulas

✱ AUTOMATIC CONTROL

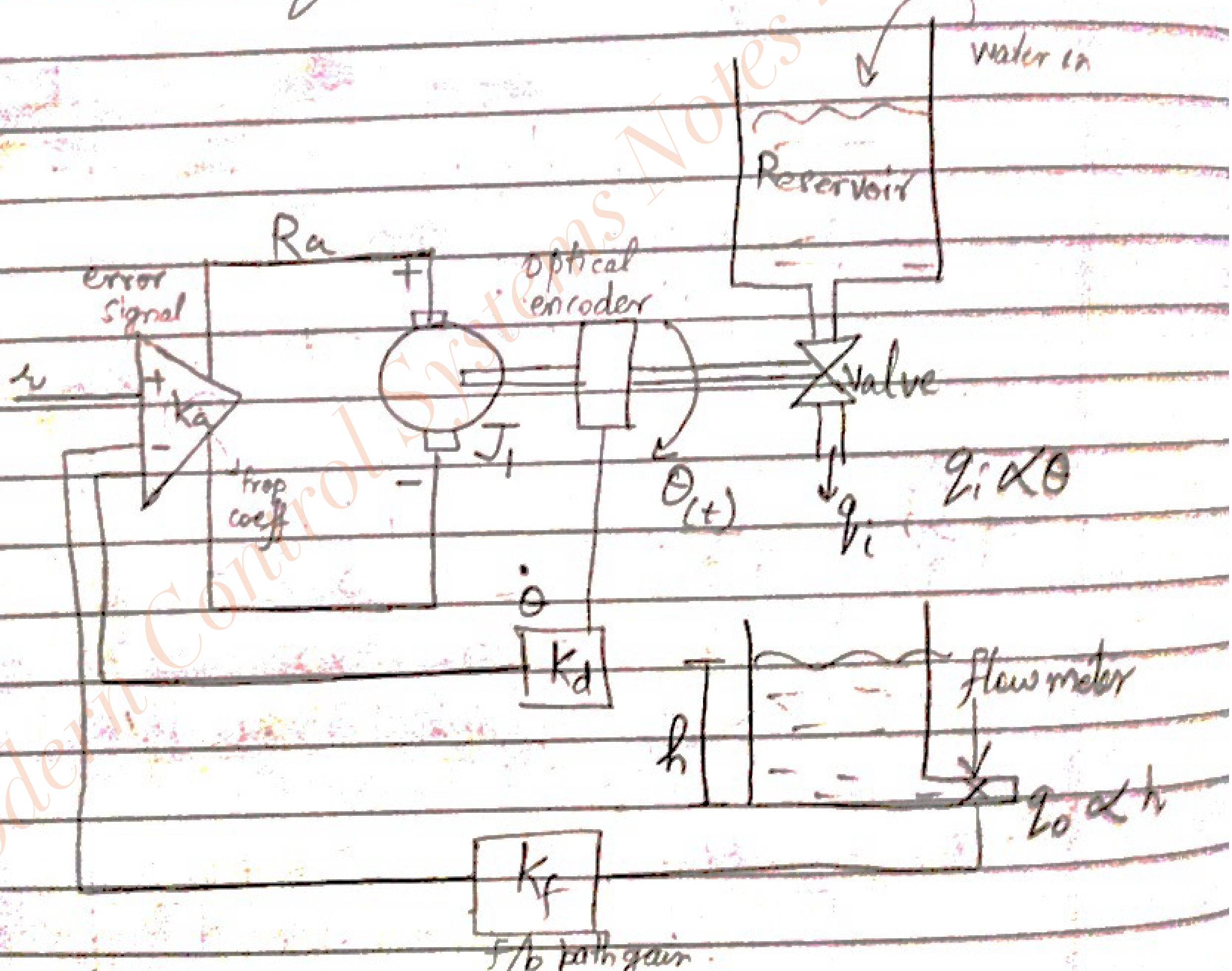
Q For the flow of control sys, prepare a signal flow graph. Identify variables & write down state variable model.

Given :- $K_a = 25$, $k_p = 1$, $k_d = 0.005$

$(K_T =) K_m = 5$, $J = 0.05$, $R_a = 1\,\Omega$

$q_i = k_q \theta$, $k_q = 8$, tank area $= A = 50\,m^2$

$q_o = k_h h$, $k_h = 225$, $k_f = 0.25$



Assuming :-

$x_1 = q_o$, $x_2 = q_i$, $x_3 = \dot{\theta}(\omega)$

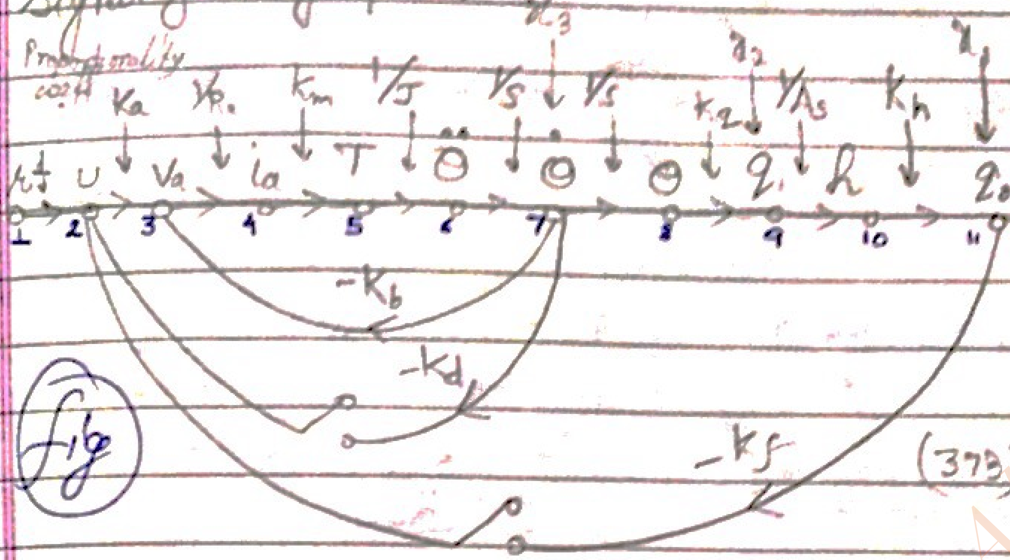
L_a = negligible

• $\frac{dq_i}{dt} = k_q \frac{d\theta}{dt} \Rightarrow \dot{x}_2$

* PID controller: It has proportional part (condition), integral part, derivative part

i/p → o/p

Signal flow graph



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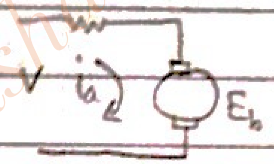
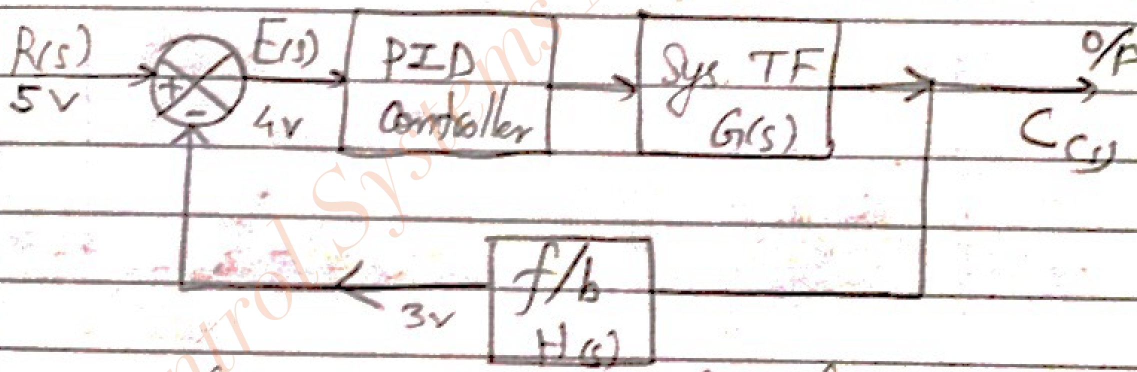


$V - E_b = I_a R_a$

K_b : Inherent in sys (doesn't come in open loop sys)

* Block diagram of SISO f/b sys



→ an electrical signal is applied to signify 5m water level supplied & o/p should be filled with 3 m tank.

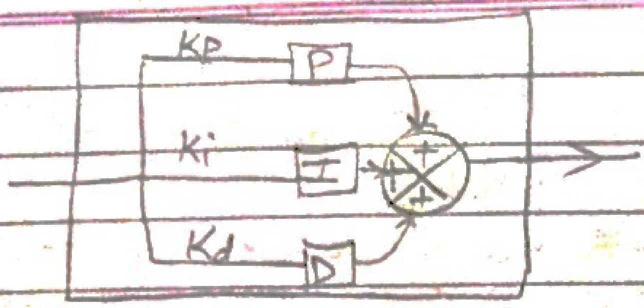
(Signal can be converted from one form to other using transducer)

Say $5m \equiv 12V$, so, $3m \equiv 6V$

Now, ∃ delay in ckt. Delay is represented as first order delay for sensor $\left(\frac{1}{Ts+1}\right)$

3v got only at steady state { If transducer is to measure 3V at o/p, it measures it as (3V) instead of (3V)

PID controller / Actuator



$$\dot{x}_1 = \frac{k_n}{A} x_2 \qquad \dot{x}_3 = (r k_a - k_b x_3)\frac{k_m}{R_a J}$$

$$\dot{x}_2 = k_q x_3 \qquad = \left(\frac{k_a k_m}{R_a J}\right) r - \left(\frac{k_b k_m}{R_a J}\right) x_3$$

$$y = q_0 = x_1$$

State eq^n in matrix form:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_n}{A} & 0 \\ 0 & 0 & k_q \\ 0 & 0 & -\frac{k_b k_m}{R_a J} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{k_a k_m}{R_a J} \end{bmatrix} [u]$$

$; u = r$

$$y = [1 \; 0 \; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} = Ax + Bu \\ y = Cx \end{bmatrix}$$

Now, consider f/b structure imposed on the closed loop state model with f/b k_d & f/b f_K.

f/b matrix:-

$$Kx = [k_f \; 0 \; k_d] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Referring to prev. (signal flow graph) with the switches closed

$$u = -Kx + r$$

Now $\dot{x} = Ax + Bu$

$$= Ax + b(-Kx + r)$$

$$= (A - bK)x + br.$$

where,

$$bK = \begin{bmatrix} 0 \\ 0 \\ \frac{k_a k_m}{R_a J} \end{bmatrix} \begin{bmatrix} k_f & 0 & k_d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{k_f k_a}{R_a J} & 0 & \frac{k_d k_a k_m}{R_a J} \end{bmatrix}$$

So,

$A - bK$ is :-

$$\begin{bmatrix} 0 & \frac{k_h}{A} & 0 \\ 0 & 0 & k_q \\ -\left(\frac{k_f k_a k_m}{R_a J}\right) & 0 & -\frac{(k_b + k_d k_a) k_m}{R_a J} \end{bmatrix}$$

Finally, state variable matrix :-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_h}{A} & 0 \\ 0 & 0 & k_q \\ -\left(\frac{k_f k_a k_m}{R_a J}\right) & 0 & -\frac{(k_b + k_d k_a) k_m}{R_a J} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{k_a k_m}{R_a J} \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In the given question, substituting given values:

$$\frac{k_d}{A} = \frac{225}{50} = 4.5$$

$$k_2 = 8$$

$$-\left[\frac{k_f k_a k_m}{R_a J}\right] = -\frac{0.25\times 25\times 5}{1\times 0.01} = -3.125\times 10^3$$

$$-\frac{(k_b + k_d k_a)k_m}{R_a J} = -\frac{(0.05 + 25\times 0.05)\times 5}{1\times 0.01} = -130$$

$$\frac{k_a k_m}{R_a J} = \frac{25\times 5}{1\times 0.01} = 12.5\times 10^3$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 4.5 & 0\\ 0 & 0 & 8\\ -3.125\times 10^3 & 0 & -130\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 12.5\times 10^3\end{bmatrix}[u]$$

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Finding TF

M1) $TF = C[sI-A]^{-1}B$.

M2) Do from signal flow graph (Done in control systems previously)

S1) Forward path :-

$P_1 = (1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12) = h\,u\,V_a\,i_a\,T\,\ddot{\theta}\,\dot{\theta}\,\theta\,q\,h\,q_o$

S2) Single loops :- (individual loops)

$(3\,4\,5\,6\,7\,3) = V_a\,i_a\,T\,\ddot{\theta}\,\dot{\theta}(-k_b)$

$(2\,3\,4\,5\,6\,7\,2) = u\,V_a\,i_a\,T\,\ddot{\theta}\,\dot{\theta}(-k_d)$

$(2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,2) = u\,V_a\,i_a\,T\,\ddot{\theta}\,\dot{\theta}\,\theta\,q\,h\,q_o(-k_f)$

S3) 2 non touching loops

none

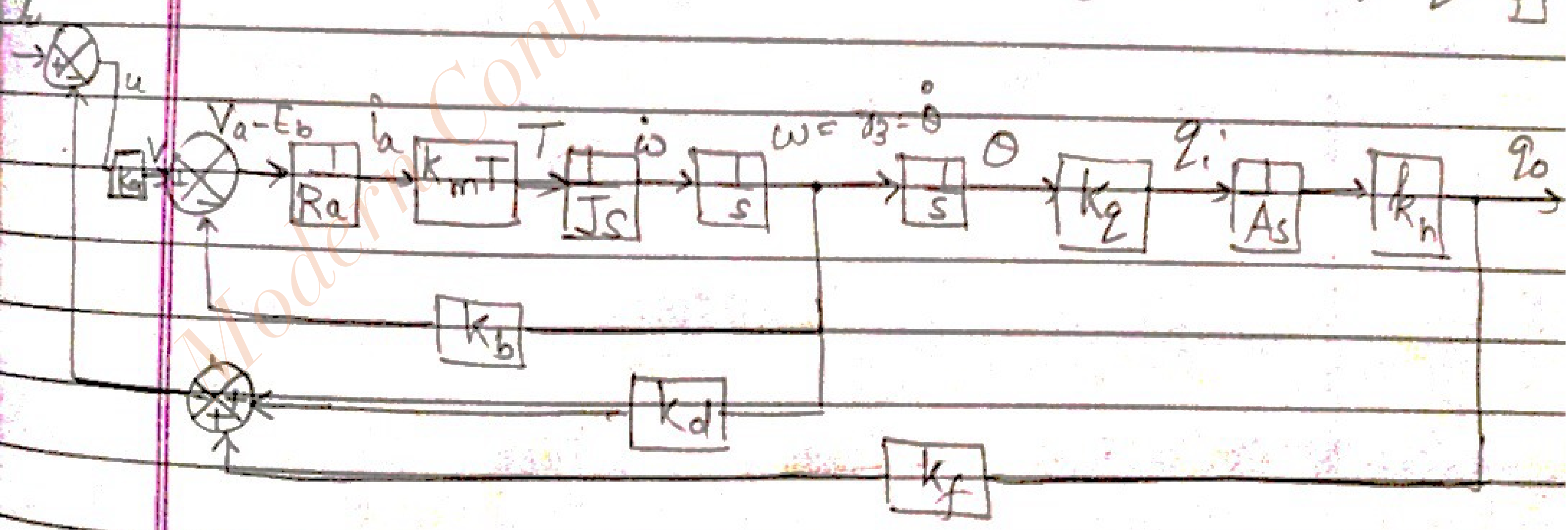
S4) $\Delta = 1 - [V_a i_a T \ddot{\theta}\dot{\theta}(-k_b) + u V_a i_a T \ddot{\theta}\dot{\theta}(-k_d) + u V_a i_a T \ddot{\theta}\dot{\theta}\theta\, q\, h\, q_o(-k_f)]$

$+ [\cancel{2 \text{ non touching}}] \to 0$

$\Delta_1 = 1$.

So, $TF = \dfrac{P_1 \Delta_1}{\Delta}$

$\Rightarrow TF = \dfrac{u V_a i_a T \ddot{\theta}\dot{\theta}\theta\, q\, h\, q_o}{1 - V_a i_a T \ddot{\theta}\dot{\theta}[-k_b - u k_d - \theta q h q_o k_f]}$



Ex Classical Control System :-
Inverted pendulum on the cart

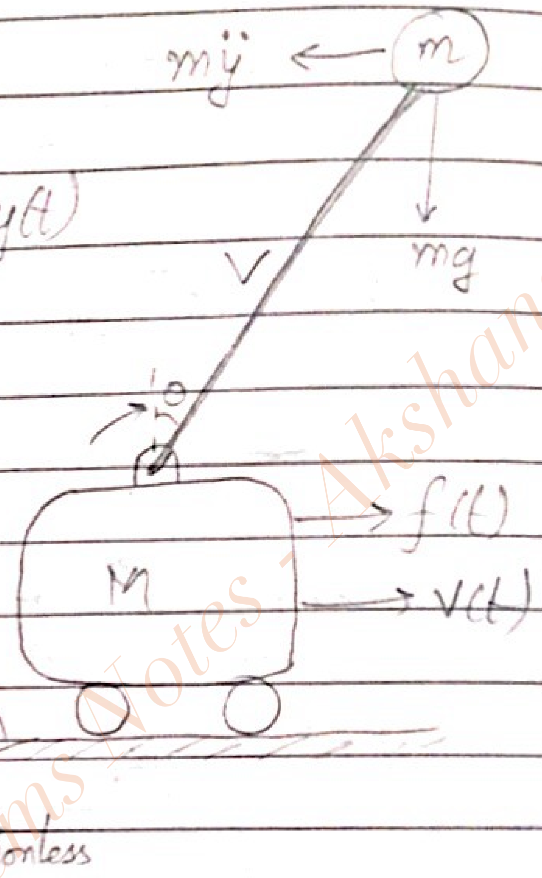
$M = 1\,Kg$

$m = 0.05\,Kg$

$L = 2\,m$.

state variables :- $\theta(t)$ & $y(t)$

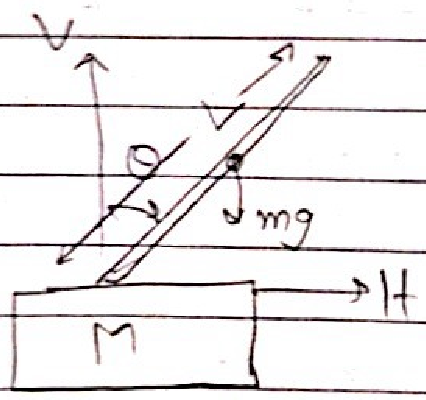
Assume :- $M >> m$,
$\theta(t)$ is very small



Aim : Make pendulum vertical
i.e., $\theta \simeq 0$

Modelling given sys

Assume :- J : MOI of pendulum w.r.t Center of gravity
mg : Force on pendulum at Center of gravity
H : Horizontal reaction force
V : Vertical reaction force



① Taking moment around center of gravity of pendulum :-

$$J\frac{d^2\theta}{dt^2} = VL\sin\theta(t) - HL\cos\theta(t) \rightarrow ①$$

Σ forces in horizontal & vertical dir^n on Pendulum

$$V - mg = m\frac{d^2}{dt^2}(L\cos\theta_{(t)}) \rightarrow (2)$$

$$H = m\frac{d^2}{dt^2}(y_{(t)} + L\sin\theta_{(t)}) \rightarrow (3)$$

Σ forces on carriage in horizontal dir^n

$$u - H = m\frac{d^2 y_{(t)}}{dt^2} \rightarrow (4)$$

We have to make $\theta \simeq 0$

$\Rightarrow \cos\theta = 1, \quad \sin\theta \simeq 0$

from (4) & (3)

$$\Rightarrow u = (M+m)\frac{d^2y}{dt^2} + mL\frac{d^2\theta}{dt^2} \rightarrow (5)$$

from (1) & (2)

$$(J + mL^2)\ddot{\theta} + mL\ddot{y} - mgL\theta = 0 \rightarrow (6)$$

Now,

$M = 1\ kg, \quad m = 0.05\ kg$

So $M + m \simeq 1\ kg\ (\simeq M)$

From (5)

$$u = M\ddot{y} + mL\ddot{\theta}$$

or $M\ddot{y} + mL\ddot{\theta} - f(t) = 0 \rightarrow (7)$

from (6)

$$mL\ddot{y} + mL^2\ddot{\theta} - mgL\theta = 0 \rightarrow (8)$$

From (7) & (8)

$$\Rightarrow \ddot{y} + l\ddot{\theta} - g\theta = 0 \rightarrow (9)$$

Take these states : $y, \dot{y}, \theta, \dot{\theta}$

State variables : $x_1 \quad x_2 \quad x_3 \quad x_4$

So, we need to find

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} ? \end{bmatrix} u$$

So, eq^{ns} (8) & (9) can be rewritten as

$$M\dot{x}_2 + ml\dot{x}_4 - f(t) = 0$$

$$\Rightarrow \dot{x}_2 = -\left(\frac{ml}{M}\right)\dot{x}_4 + \frac{1}{M}f(t) \longrightarrow (10)$$

$$\& \quad \dot{x}_2 + l\dot{x}_4 - gx_3 = 0.$$

$$\Rightarrow \dot{x}_4 = -\frac{1}{l}\dot{x}_2 + \frac{g}{l}x_3 \longrightarrow (11)$$

from (10) & (11)

$$M\dot{x}_2 + ml\left[-\frac{1}{l}\dot{x}_2 + \frac{g}{l}x_3\right] - f(t) = 0$$

$$(M-m)\dot{x}_2 + mgx_3 - f(t) = 0.$$

$$M >> m \Rightarrow M\dot{x}_2 + mgx_3 - f(t) = 0$$

from (11) & (10) again

$$\Rightarrow \left(l - \frac{ml}{M}\right)\dot{x}_4 - gx_3 + \left(\frac{1}{M}\right)f(t) = 0$$

$\Rightarrow (M - m)\, l \dot{x}_4 - Mg x_3 + f(t) = 0$

$\Rightarrow M l \dot{x}_4 - Mg x_3 + f(t) = 0.$

So, state variable eq^ns are:-

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\left(\frac{mg}{M}\right)x_3 + \left(\frac{1}{M}\right)f(t)$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \left(\frac{g}{l}\right)x_3 - \left(\frac{1}{Ml}\right)f(t)$$

Matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -\frac{mg}{M} & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & \frac{g}{l} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 1/M\\ 0\\ -1/Ml\end{bmatrix}\left[f(t)\right]$$

evaluating coeff. $\frac{mg}{M} = 0.49$, $\frac{g}{l} = 49$; $\frac{1}{M} = 1$

$\frac{1}{Ml} = 5$.

$$\Rightarrow A = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -0.49 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 49 & 0\end{bmatrix}; \quad B = \begin{bmatrix}0\\ 1\\ 0\\ -5\end{bmatrix}$$

$$C = [1 \; 0 \; 0]$$

* linear sys :- System satisfying superpos^n, homogeneity prop.

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* State space represent^n using Phase variables

LCT sys → linear Continuous Time

$$y^n + a_1 y^{n-1} + \dots + a_{n-1}\dot{y} + a_n y = b_0 u^m + b_1 u^{m-1} + \dots + b_{m-1}\dot{u} + b_m u$$

Taking LT, both sides

$$\Rightarrow s^n Y(s) + a_1 s^{n-1} Y(s) + \dots + a_n Y(s) = b_0 s^m U(s) + b_1 s^{m-1} U(s) + \dots + b_m U(s)$$

$$\Rightarrow (s^n + a_1 s^{n-1} + \dots + a_n) Y(s) = (b_0 s^m + b_1 s^{m-1} + \dots + b_m) U(s)$$

$$\Rightarrow TF = T(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n}$$

(numerator gives zeros, denominator gives poles) ①

→ order of sys = n.

→ Poles, $n \geq$ Zeros, m

For simple case with no zeros,

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b}{s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n} \rightarrow ②$$

Solving sys. ② i.e, no zeroes.

Now, assuming state variables

Let : First o/p variable be first state var.

So, $x_1 = y$

$x_2 = \dot{y}$

$x_n = y^{(n-1)} \rightarrow \frac{d^{n-1}}{dt^{n-1}} y$

So, we got $\dot{x}_1 = x_2$

$$\dot{x}_2 = x_3$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n$$

So we have $\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \dots - a_1 x_n + bu$

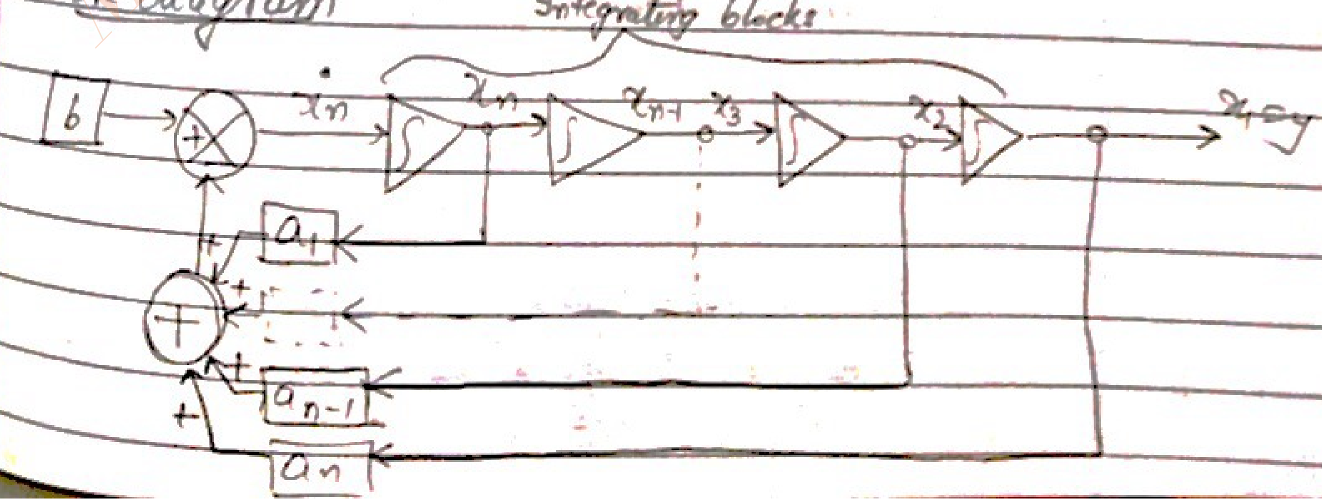
So, directly, state space representⁿ i.e. :-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & -a_1 \end{bmatrix}}_{\text{COMPANION form}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u$$

~~CAN~~ COMPANION form
(off diagonal elements are 1, rest all elements are 0 → except last row)

$$\& \; y = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Block diagram



Page

Solving sys ①: m zeros, n poles

Considering a 3rd order sys.

$$TF: T(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{(1) s^3 + a_1 s^2 + a_2 s + a_3}$$

make it 1 always when representing any order sys.

$\div s^3$ & taking common

$$\Rightarrow T(s) = \frac{b_0 + b_1/s + b_2/s^2 + b_3/s^3}{1 - (-a_1/s - a_2/s^2 - a_3/s^3)}$$

By Mason's formula, Finding Gain :-

$$T(s) = \frac{P_1 \Delta_1 + P_2 \Delta_2 + \cdots}{\Delta}$$

(formula used in Signal flow graph)

$$\Delta = 1 - (\Sigma \text{ Individual loops}) + (\Sigma \text{ 2 non touching}) - (\Sigma \text{ 3 non touching}) + \cdots$$

So, for given TF,

$$\Delta = 1 - \Sigma(\text{Individual loop gains})$$

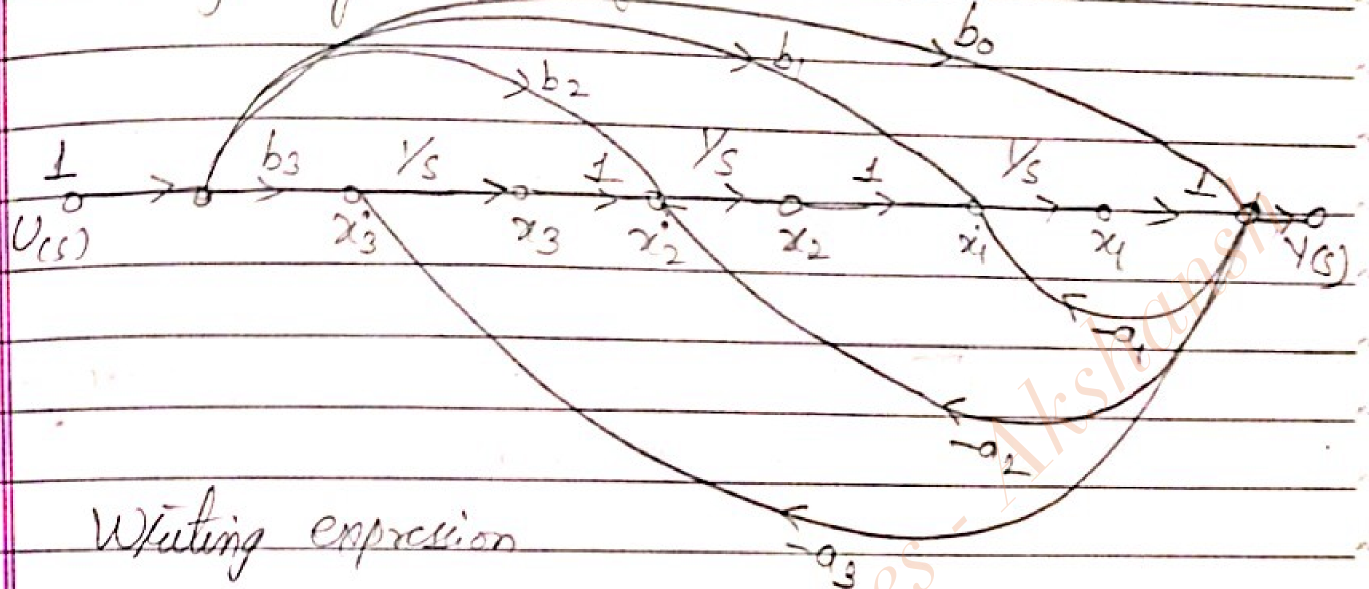
So, individual loop gains $= \frac{-a_1}{s}, \frac{-a_2}{s^2}, \frac{-a_3}{s^3}$

Now, $\Delta_1, \Delta_2, \Delta_3 = 1$,

$P_0 = b_0$, $P_1 = \frac{b_1}{s}$, $P_2 = \frac{b_2}{s^2}$, $P_3 = \frac{b_3}{s^3}$

We have $y = x_1 + b_0 u$.

Making signal flow graph :-



Writing expression

$$\dot{x}_1 = -a_1(x_1 + b_0 u) + x_2 + b_1 u$$
$$= -a_1 x_1 + x_2 + (b_1 - a_1 b_0)u$$
$$\dot{x}_2 = -a_2 x_1 + x_3 + (b_2 - a_2 b_0)u$$
$$\dot{x}_3 = b_3 u - a_3 y = b_3 u - a_3(x_1 + b_0 u) = -a_3 x_1 + (b_3 - a_3 b_0)u$$

done like KCL in circuits

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 & -a_1 b_0 \\ b_2 & -a_2 b_0 \\ b_3 & -a_3 b_0 \end{bmatrix} [u]$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 u$$

☆ Initial cond^{ns} given can be put

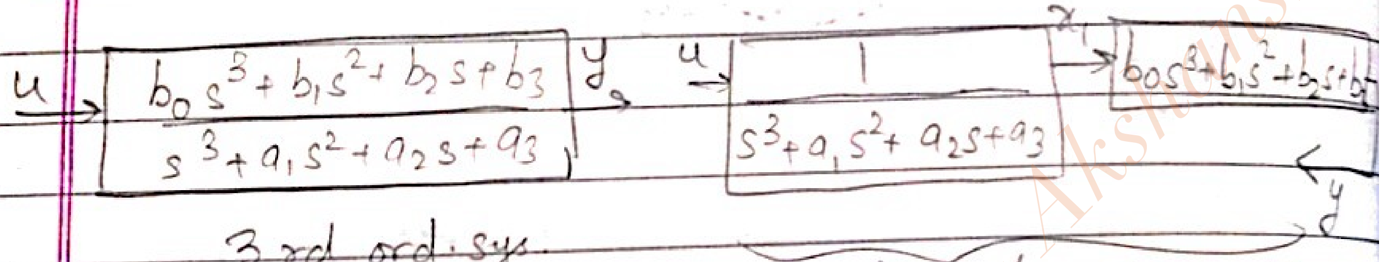
- - - can be extended to any order.

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & a_1 & 1 \end{bmatrix} \begin{bmatrix} y(0) \\ \dot{y}(0) \\ \ddot{y}(0) \end{bmatrix} + \begin{bmatrix} -b_0 & 0 & 0 \\ -b_1 & -b_0 & 0 \\ -b_2 & -b_1 & -b_0 \end{bmatrix} \begin{bmatrix} u(0) \\ \dot{u}(0) \\ \ddot{u}(0) \end{bmatrix}$$

Taking numerator & den. poly. separately

$$\frac{X_1(s)}{U(s)} = \frac{1}{s^3 + a_1 s^3 + a_2 s + a_3}$$

$$\frac{Y(s)}{X_1(s)} = b_0 s^3 + b_1 s^2 + b_2 s + b_3$$



3 rd ord. sys.

Alternate representⁿ of sys.

For zero no. of zero's → 3rd ord

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Now, from TF :- $\dddot{x}_1 \to \dot{x}_3$, $\ddot{x}_1 \to x_3$, $\dot{x}_1 \to x_2$

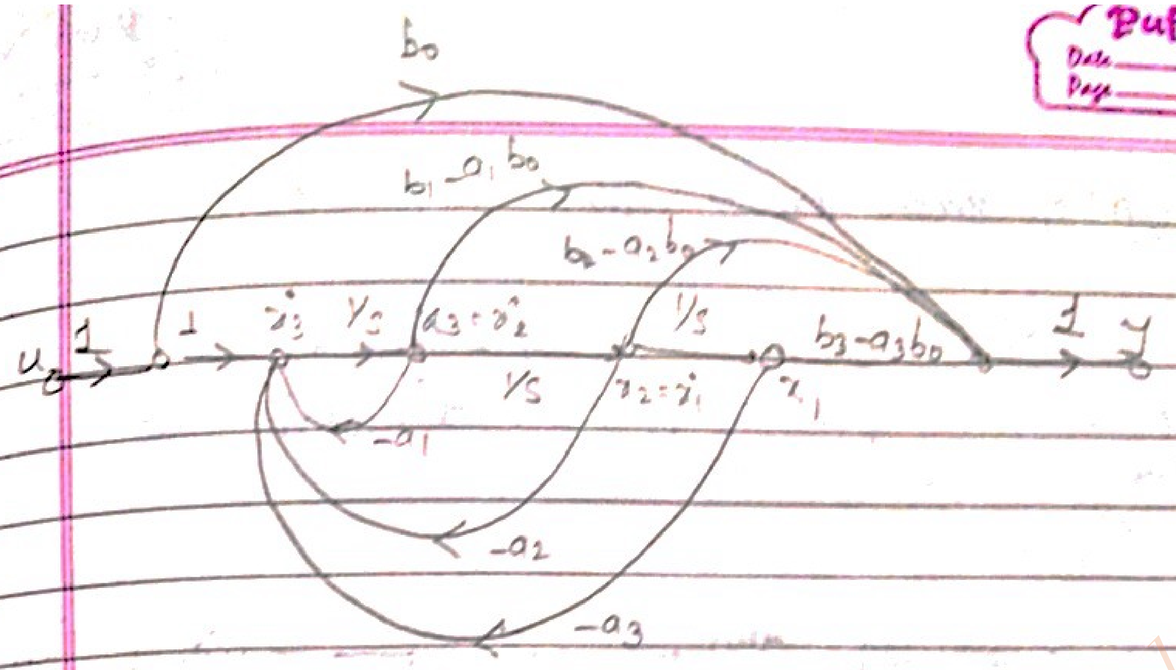
$$y = b_0 \dddot{x}_1 + b_1 \ddot{x}_1 + b_2 \dot{x}_1 + b_3 x_1$$

$$= b_0(-a_3 x_1 - a_2 x_2 - a_1 x_3 + u) + b_1 x_3 + b_2 x_2 + b_3 x_1$$

$$= (b_3 - a_3 b_0) x_1 + (b_2 - a_2 b_0) x_2 + (b_1 - a_1 b_0) x_3 + b_0 u$$

Finding C matrix :-

$$y = \begin{bmatrix} (b_3 - a_3 b_0) & (b_2 - a_2 b_0) & (b_1 - a_1 b_0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 u$$



* State space represent^n using cannonical variables

$$y(s) = T(s) = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{\underbrace{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}}$$

nth ord sys $= \underbrace{1 + 1 + \dots 1}$

Sum of n single order systems

$$= b_0 + \sum_{i=1}^{n} \frac{c_i}{s - \lambda_i}$$

c_i → The residues

λ_i → The poles

State space matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

↳ Cannonical form

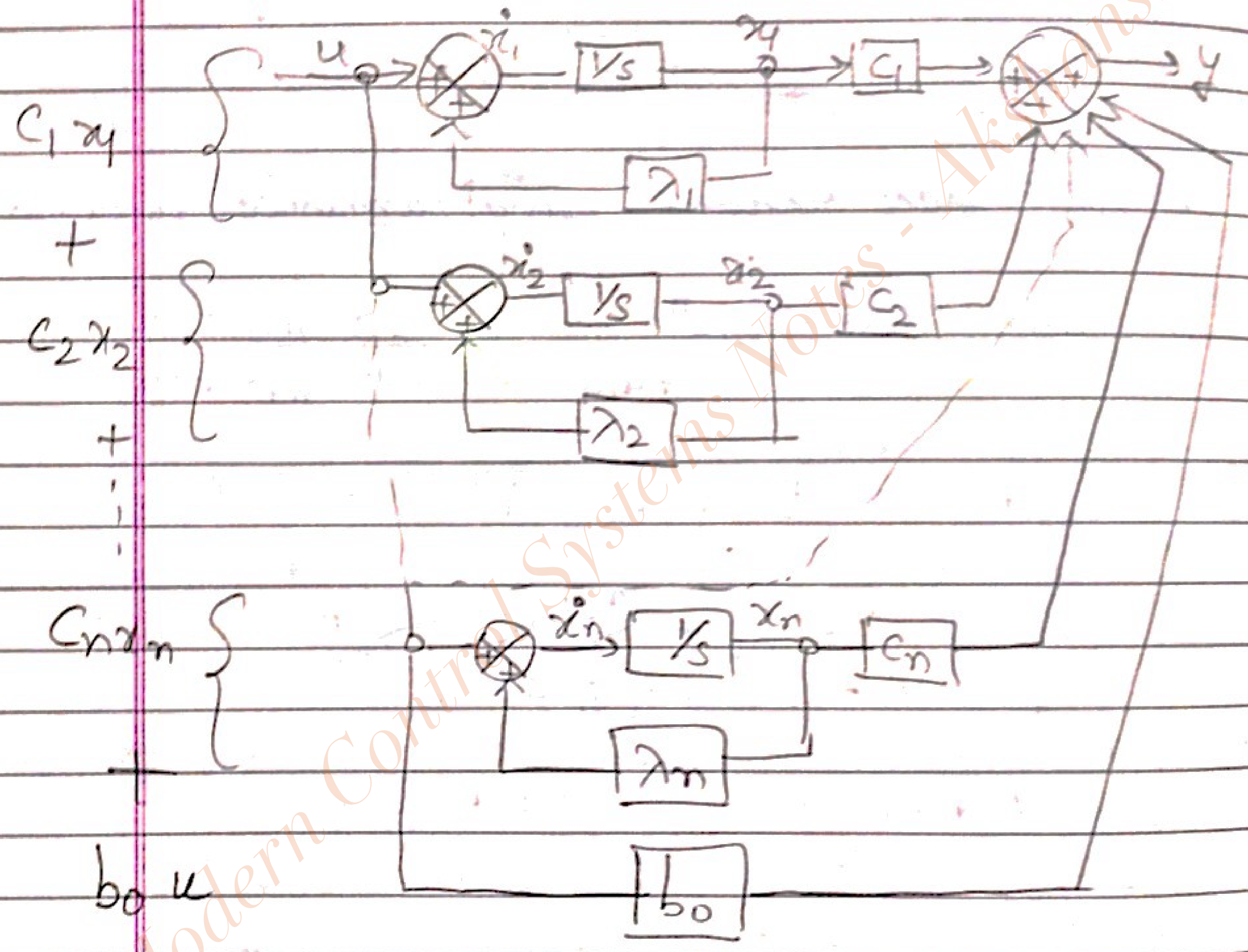
$\lambda_1, \lambda_2, \dots \lambda_n$: Eigen values

$$y = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

o/p of each integrator

↳ $\dot{x}_i = \lambda_i x_i + u$; $i = 1, 2, \dots n$

o/p → $y = c_1 x + c_2 x_2 + \dots + c_n x_n + b_0 u$
$y(t)$



Block diagram of Cannonical state model.

* i/p matrix & o/p matrix can be interchanged
(next pg)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Q. Consider a D.E of a control sys:-

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = \dddot{u} + 8\ddot{u} + 17\dot{u} + 8u$$

Taking Laplace Transform

$$\Rightarrow s^3 Y(s) + 6s^2 Y(s) + 11s Y(s) + 6Y(s)$$
$$= s^3 U(s) + 8s^2 U(s) + 17s U(s) + 8U(s)$$

$$\Rightarrow (s^3 + 6s^2 + 11s + 6) Y(s)$$
$$= (s^3 + 8s^2 + 17s + 8) U(s)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{s^3 + 8s^2 + 17s + 8}{s^3 + 6s^2 + 11s + 6}.$$

Writing Cannonical form :

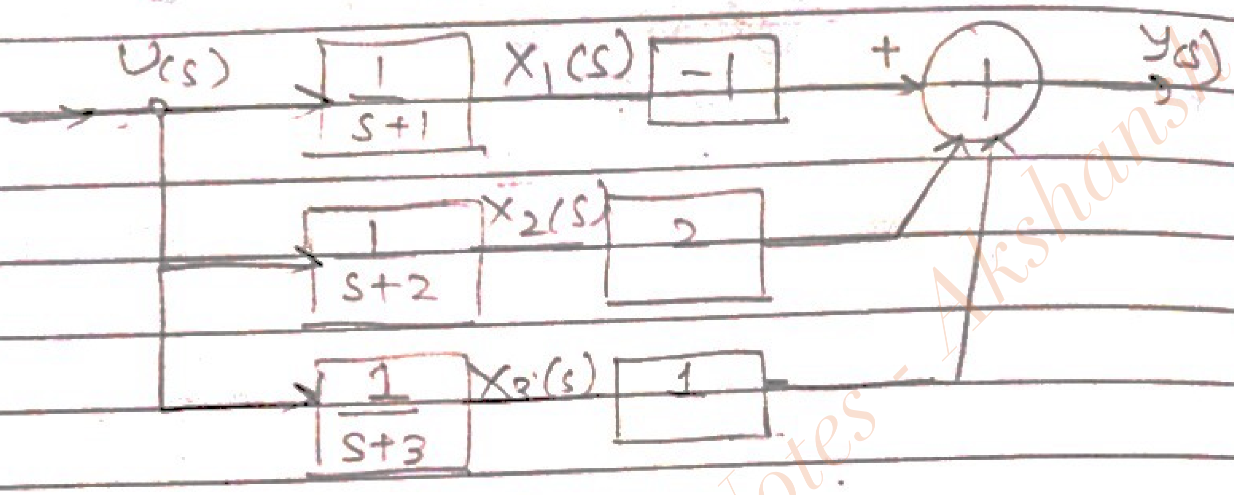
Idea:- Make partial fractions.

$$\frac{Y(s)}{U(s)} = \frac{s^3 + 8s^2 + 17s + 8}{s^3 + 6s^2 + 5s + 6s + 6} = \frac{s^3 + 8s^2 + 17s + 8}{(s+2)(s+3)(s+1)}$$

eq^n (1) $\Rightarrow \dfrac{Y(s)}{U(s)} = 1 - \dfrac{1}{s+1} + \dfrac{2}{s+2} + \dfrac{1}{s+3}$

here eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ are $(-1, -2, -3)$

Block diagram



☆ Finding state space model if partial fraction is given:-

eg:- eq^n (2) $\dfrac{sY(s)}{U(s)} = s+2 + \dfrac{1}{s+1} - \dfrac{4}{s+2} - \dfrac{3}{s+3}$

eq^n (3) $\dfrac{s^2 Y(s)}{U(s)} = s^2 + 2s - 6 - \dfrac{1}{s+1} + \dfrac{8}{s+2} + \dfrac{9}{s+3}$

From (2), taking $(s+2)$ on LHS

$\Rightarrow \dfrac{sY(s) - sU(s) - 2U(s)}{U(s)} = \dfrac{1}{s+1} + (-4)\dfrac{1}{s+2} + (-3)\dfrac{1}{s+3}$

&

$\dfrac{s^2 Y(s) - s^2 U(s) - 2s(U(s)) + 6U(s)}{U(s)} = \dfrac{-1}{s+1} + (8)\dfrac{1}{s+2} + (9)\dfrac{1}{s+3}$

Now matrix form looks like
(from eq^{ns} ①, ②, ③)

$$\Rightarrow \begin{bmatrix} \dfrac{Y(s)-U(s)}{U(s)} \\[2mm] \dfrac{sY(s)-sU(s)-2U(s)}{U(s)} \\[2mm] \dfrac{s^2Y(s)-s^2U(s)-2U(s)+6U(s)}{U(s)} \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 1 & -4 & -3 \\ -1 & 8 & 9 \end{bmatrix} \begin{bmatrix} \dfrac{1}{s+1} \\[2mm] \dfrac{1}{s+2} \\[2mm] \dfrac{1}{s+3} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 1 \\ 1 & -4 & -3 \\ -1 & 8 & 9 \end{bmatrix} \begin{bmatrix} \dfrac{X_1(s)}{U(s)} \\[2mm] \dfrac{X_2(s)}{U(s)} \\[2mm] \dfrac{X_3(s)}{U(s)} \end{bmatrix}$$

Taking L^{-1}

$$\Rightarrow \underbrace{\begin{bmatrix} y-u \\ \dot{y}-\dot{u}-2u \\ \ddot{y}-\ddot{u}-2\dot{u}+6u \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 1 & -4 & -3 \\ -1 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

Can be used to find initial cond^{ns} of the defined state variables.

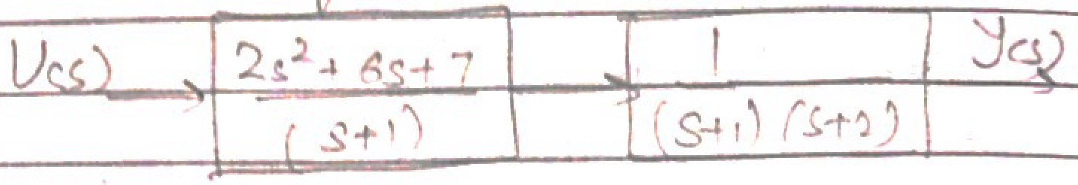
how

$$U(s) \longrightarrow \boxed{\frac{1}{s+1}} \longrightarrow X_1(s)$$

i.e $\dfrac{X_1(s)}{U(s)} = \dfrac{1}{s+1}$

(||ly, others)

ex Consider a controller and automatic wheelchair. A TF is given. Find state space model



Now, $\frac{Y(s)}{U(s)} = \frac{2s^2+6s+7}{(s+1)^2(s+2)}$

Using Partial fraction

$\Rightarrow$

$$\frac{Y(s)}{U(s)} = \frac{-1}{s+1} + \frac{9}{(s+1)^2} + \frac{3}{(s+2)} \longrightarrow (1)$$

Defining state var.

$$\frac{X_1(s)}{X_2(s)} = \frac{1}{s+1} \quad \text{or} \quad \frac{X_2(s)}{s+1} = X_1(s)$$

$$\Rightarrow x_2 = \dot{x}_1 + x_1 \quad \Rightarrow \dot{x}_1 = -x_1 + x_2$$

& $\frac{U(s)}{s+1} = X_2(s) \Rightarrow u = \dot{x}_2 + x_2 \Rightarrow \dot{x}_2 = -x_2 + u.$

& $\frac{U(s)}{s+2} = X_3(s) \Rightarrow u = \dot{x}_3 + 2x_3 \Rightarrow \underbrace{\dot{x}_3 = -2x_3 + u}_{\text{how?}}$

consider :- $X_2(s) = X_1(s)(s+1)$

or $X_2 = X_1(s+1)$

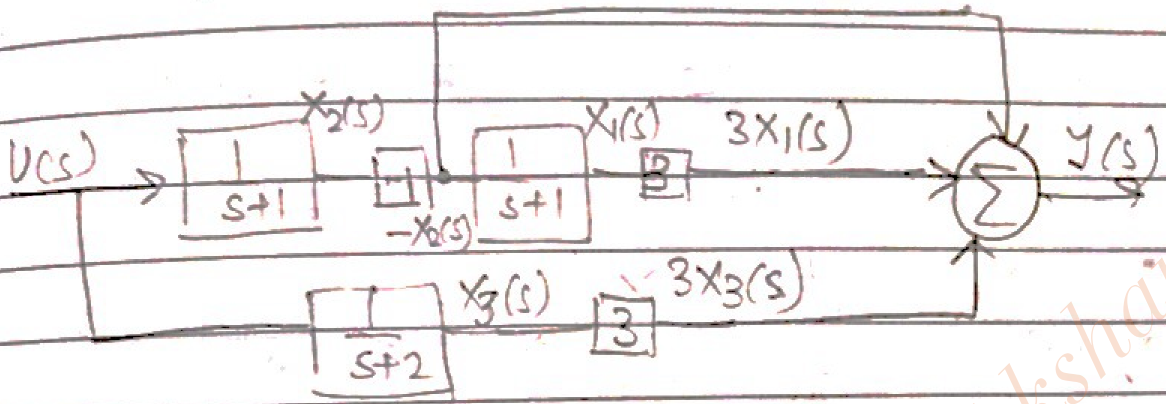
$\mathcal{L}^{-1}$

$\Rightarrow x_2 = \mathcal{L}^{-1}(sX_1 + X_1)$

$\Rightarrow x_2 = \dot{x}_1 + x_1$

Ily others.

Block diagram



Now, using eqⁿ (1)

$$Y(s) = \frac{-U(s)}{s+1} + (3)\frac{U(s)}{(s+1)^2} + 3\frac{U(s)}{(s+2)}$$

$$\Rightarrow Y(s) = -X_{2}(s) + 3X_1(s) + 3X_3(s)$$

or $y = -x_2 + 3x_1 + 3x_3$

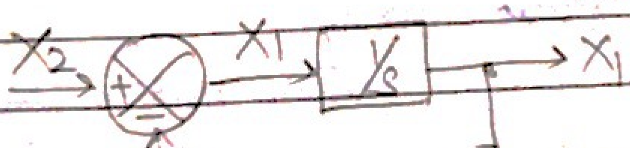
Solving for block diagram:-

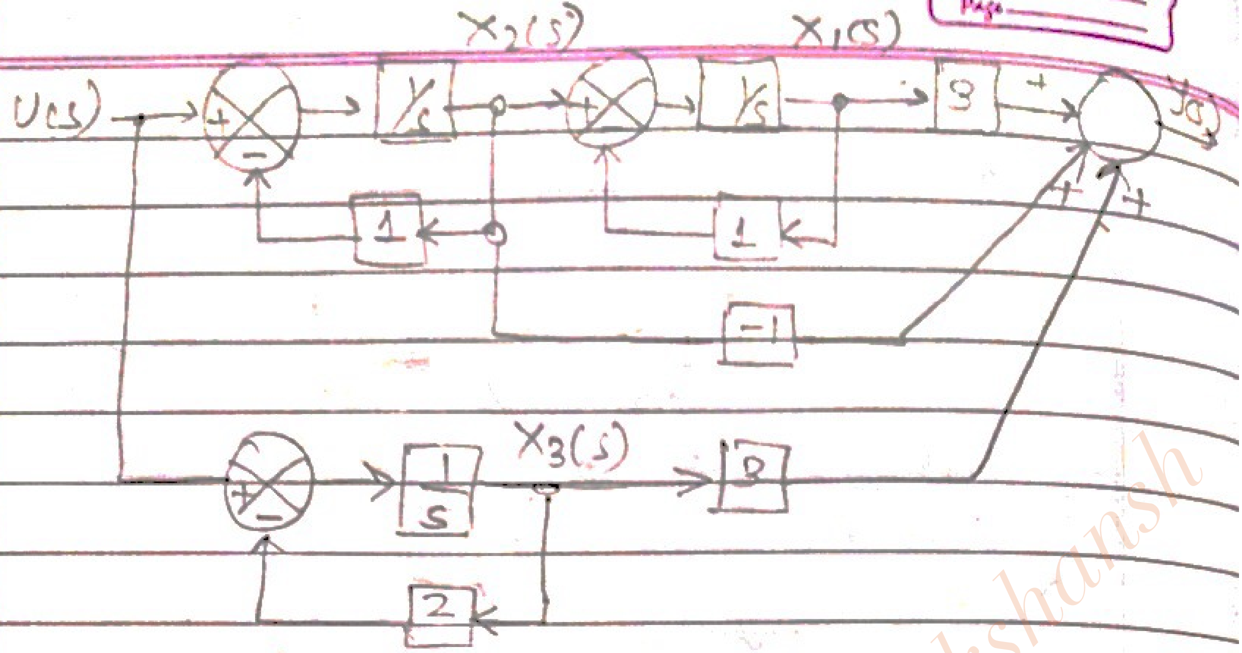
We know $\frac{X_1(s)}{X_2(s)} = \frac{1}{s+1}$

$$= \frac{1/s}{1+1/s} \quad \left(\equiv \frac{G(s)}{1+G(s)}\right)$$

Rule 9 → Control Sys.

So,





Matrix form of time domain equations :-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} u \end{bmatrix}.$$

$$y = \begin{bmatrix} 3 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Jordon Block : If off diagonal elements are non zero.

☆ ~~Giv~~ Given:-

System matrix = A

I/p matrix = B

O/p matrix = C.

Finding TF : $\frac{Y(s)}{U(s)} = C\left[sI - A\right]^{-1} B + d.$

$$= C\left[\frac{adj(sI-A)}{det(sI-A)}\right] B + d$$

$\Rightarrow TF = \dfrac{C \times adj(sI-A) \times B}{\det(sI-A)} + d.$

Now, we know,

for a TF, denominator is called Characteristic eq^n.

Roots of char. eq^n gives Eigen values

Thus here, the denominator is $\det(sI-A)$

So, the eq^n $|sI-A| = 0$ will give the POLE LOCATIONS (eigenvalues) for given TF

DIAGONALISATION

Consider an n^{th} order multi i/p, multi o/p state model.

$$\dot{x} = Ax + Bu \; ; \; y = Cx + Du.$$

Aim: Make A, a diagonal matrix. So, pole loc^{ns} will be diagonal elements.

Now, assume A is non diagonal.

Now, define a new state vector $x = Mv$ [M is $n \times n$ non singular const]

$$\therefore \dot{v} = M^{-1}[AMv] + M^{-1}[Bu]$$

$$\& \; y = CMv + Du.$$

Λ : Tilda

If M can be selected s.t. $M^{-1}AM$ is diagonal matrix, M is called Diagonalising matrix or Model Matrix.

$$\text{Let } M^{-1}AM = \Lambda$$

$$\therefore \dot{v} = \Lambda v + \tilde{B} u$$

$$\& \; y = \tilde{C} V + Du \; ; \; \tilde{B} = M^{-1}B$$

$$\tilde{C} = CM$$

- Eigenvalues and Eigenvectors

Assume a transform^n matrix A with $Ax = y$

$$\underbrace{A}_{n\times n} \underbrace{x}_{n\times 1} = \underbrace{y}_{n\times 1}$$

Whether $\exists$ a vector x, s.t. matrix operator A transforms into a vector λx (λ : constt) i.e. with same dir^n as vector x

The a sol^n of $Ax = \lambda x$

So, sol^n is $(\lambda I - A)x = 0$ i.e. matrix eq^n or characteristic eq^n

The set of homogeneous eq^{ns} have sol^{ns} if & only if

$$\underbrace{|\lambda I - A| = 0}_{\text{Eigenvalues of matrix}} \quad \text{Pole Loc}^{ns}$$

$(\equiv |sI - A| = 0$ in Control Sys$)$

Hence, proved that pole loc^{ns} = Eigenvalues.

$(\lambda I - A) = 0$ can be expressed as:-

$$q(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \text{---} + a_n = 0,$$

The values of λ for the eq^n is called Eigenvalues of matrix A and the eq^n (characteristic eq^n) corresponding to matrix A

for a TF : $q(s) = |sI - A| = 0$ will give poles.

$\therefore$ Poles are eigenvalues.

Hence, a state model is stable, when real part of Eigenvalues are negative.

Now, for $\lambda = \lambda_i$

So, $(\lambda_i I - A)x = 0$; let $x = m_i$ be the sol^n of this eq^n

So, sol^n m_i is called eigenvector of A associated with eigenvalue λ_i

The sol^n depends on rank of matrix $[\lambda I - A]$

If rank is r, $\exists$ $n-r$ independent sol^n (eigenvectors).

If eigenvalues are distinct, rank $= n-1$ & only one independent eigenvector for one λ_i.

This vector may be obtained by taking cofactor of matrix $(\lambda_i I - A)$ along any row.

$$m_i = \begin{bmatrix} C_{k1} \\ C_{k2} \\ \vdots \\ C_{kn} \end{bmatrix} \quad ; k = 1, 2, \ldots n$$

C_{k_i} : cofactors of matrix $(\lambda I - A)$

Let $m_1, m_2, \ldots, m_n$ be the eigenvectors corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$

Then,

$$AM = A[m_1 \vdots m_2 \vdots m_3 \vdots \ldots m_n]$$
$$= [Am_1 \vdots Am_2 \vdots Am_3 \ldots Am_n]$$
$$= [\lambda_1 m_1 \vdots \lambda_2 m_2 \vdots \lambda_3 m_3 \ldots \lambda_n m_n]$$
$$= M \Lambda$$

→ capital λ

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ & 0 & \lambda_3 & \ldots & 0 \\ & 0 & & & 0 \\ 0 & 0 & & \ldots & \lambda_n \end{bmatrix}$$

$$= M^{-1} A M$$

i.e., eigenvector matrix M is the diagonalising matrix or Modal matrix M of A.

Λ & A will have same invariant eigenvalue.

If $A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{bmatrix}$

i.e :- Companion form (No zeroes)

Then, modal matrix can be shown as:

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & & \lambda_n^2 \\ \lambda_1^3 & \lambda_2^3 & & \lambda_n^3 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & & \lambda_n^{n-1} \end{bmatrix}$$

$\rightarrow$ VANDER MONDE MATRIX.

eg Consider the following matrix :-

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$$

Finding characteristic eq^n :- $(\lambda I - A) = 0$

$$\Rightarrow \begin{bmatrix} \lambda & -1 & 0 \\ -3 & \lambda & -2 \\ 12 & 7 & \lambda+6 \end{bmatrix} = 0.$$

Now, finding sol^n for homogeneous eq^n

$$\Rightarrow \lambda(\lambda^2+6\lambda+14)+1(-3\lambda-18+24) = 0.$$

$$\Rightarrow (\lambda+1)(\lambda+2)(\lambda+3) = 0.$$

So, eigenvalues of matrix A are :-

$\lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3$

Now, finding m_1 associated with $\lambda_1 = -1$ obtained by cofactors of matrix

$$(\lambda_1 I - A) = \begin{bmatrix} -1 & -1 & 0 \\ -3 & -1 & -2 \\ 12 & 7 & 5 \end{bmatrix}$$

$C_{11} = 9$

$C_{12} = -9$

$C_{13} = -9$

So, $m_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ -9 \end{bmatrix}$

or $m_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

m_i can also be obtained from solⁿ of homogeneous eqⁿ corresponding to matrix $(\lambda I - A)x = 0$; i.e.,

$$-x_1 + x_2 = 0$$
$$-3x_1 - x_2 - 2x_3 = 0$$
$$12x_1 + 7x_2 + 5x_3 = 0.$$

Choose $x_1 = 1,\ x_2 = -1,\ x_3 = -1$ we get the same solⁿ as got in cofactors for m_1.

Now, finding $m_2 \big|_{\lambda_2 = -2}$.

$m_2 = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$

$m_3 \big|_{\lambda_3 = -3}$; $m_3 = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$

Total matrix M is obtained as:-

$$M = [m_1 \mid m_2 \mid m_3] = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ 1 & 1 & 3 \end{bmatrix}$$

It can now be verified that

$$\Lambda = [M^{-1}AM] \rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

eg(2): Consider the matrix :-

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

If eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 3$

Find eigenvector for λ_1

$$(\lambda_1 I - A) = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix}$$

$$m_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ all zeroes doesn't work.

looking at R_2 instead (for m_1)

$$m_1 = \begin{bmatrix} C_{21} \\ C_{22} \\ C_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}$$

Eigenvector for $\lambda_2 = 3$

$$(\lambda_2 I - A) = \begin{bmatrix} \lambda_2 - 4 & -1 & 2 \\ -1 & \lambda_2 & -2 \\ -1 & 1 & \lambda_2 - 3 \end{bmatrix}$$

Note: rank of 3×3 matrix $(\lambda_2 I - A) = 2$

∴ independent eigenvector with $\lambda = 3$ can be obtained.

$$m_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} \lambda_2(\lambda_2 - 3) + 2 \\ (\lambda_2 - 3) + 2 \\ -1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Now, vector m_3 for modal matrix

$$M = [m_1 \mid m_2 \mid m_3]$$

generated from independent eigenvector m_2 as follows:-

$$m_3 = \begin{bmatrix} \frac{d}{d\lambda_2} C_{11} \\ \frac{d}{d\lambda_2} C_{12} \\ \frac{d}{d\lambda_3} C_{13} \end{bmatrix} = \begin{bmatrix} 2\lambda - 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

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Solution of a state eq^n:-

eq^n :- $\frac{dx}{dt} = ax$; $x(0) = x_0$.

eq^n has sol^n:-

$$x(t) = e^{at} x_0.$$

$$= \underbrace{\left(1 + at + \frac{1}{2!}a^2t^2 + \dots + \frac{1}{i!}a^it^i + \dots\right)}_{e^{at}} x_0$$

Now consider the state eq^n:-

$$\dot{x}(t) = A\,x(t) \; ; \; x(0) = x_0.$$

(Note :- $u = 0 \Rightarrow$ homogeneous eq^n)

Assume sol^n as :-

$$x(t) = a_0 + a_1 t + \dots a_i t^i + \dots -$$

Substituting $x(t)$ in given sys:-

$$\Rightarrow \dot{x}(t) = a_1 + 2a_2 t + \dots = A(a_0 + a_1 t + \dots)$$

Comparing

$$\Rightarrow a_1 = A\,a_0$$

$$a_2 = \frac{1}{2} A a_1 = \frac{1}{2!} A^2 a_0.$$

$$\vdots$$

$$a_i = \frac{1}{i!} A^i a_0$$

Assuming $x(t=0) = x_0$, we find

$$a_0 = x_0$$

★ matrix A : describes dynamics of sys

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Sol^n, thus is :- initial state

$$x(t) = \left(I + At + \frac{1}{2!}A^2t^2 + \dots + \frac{1}{i!}A^it^i + \dots\right)x_0 = e^{At}x_0$$

$$\Rightarrow e^{At} = I + At + \frac{1}{2!}A^2t^2 + \dots + \frac{1}{i!}A^it^i + \dots$$

Basically, to find time domain response for our sys in matrix form, we need to evaluate e^{At}, (where A is a matrix).

Notation :- We are going from x_0 at $t=0$ to $x(t)$ at $t=t$. The trans^n goes with the matrix e^{At}

e^{At} : called as **State Trans^n Matrix** denoted by $\phi(t)$

Considering
Non homogeneous eq^n

$$\dot{x}(t) = Ax(t) + Bu(t) \; ; \; x(0) = x_0$$

rewriting

$$\Rightarrow \dot{x}(t) - Ax(t) = Bu(t)$$

× both sides by e^{-At}.

$$\Rightarrow e^{-At}\left[\dot{x}(t) - Ax(t)\right] = \frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}u(t)$$

Integrating both sides from 0 to t

$$\Rightarrow e^{-At}x(t)\Big|_0^t = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$\Rightarrow e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$\times\ e^{At}$

$$\Rightarrow x(t) = \underbrace{e^{At}x(0)}_{\text{homogeneous sol}^n} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{forced sol}^n}$$

due to initial cond^n or Transient response

→ due to applied i/p or steady state response

Complete sol^n wrt the dynamic matrix A

Generalising response from $t = t_0$ to $t = t$.

$$\Rightarrow x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-t_0)}Bu(\tau)d\tau.$$

* Properties of state trans^n matrix ($\phi(t)$)

1) $\phi(t-t_0) = e^{A(t-t_0)}$
2) $\phi(0) = e^{A(0)} = I$
3) $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$

or $\phi^{-1}(t) = \phi(-t)$

4) $\phi(t_1+t_2) = e^{A(t_1+t_2)} = e^{At_1}\cdot e^{At_2}$

$= \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$

* Comput^n of State Trans^n Matrix

Consider unforced sys:-

$$\dot{x} = Ax$$

LT

$$\Rightarrow sX(s) - x(0) = AX(s)$$

or

$$[sI - A]X(s) = x(0)$$

$$\Rightarrow X(s) = [sI - A]^{-1}x(0)$$

$\mathcal{L}^{-1}$

$$\Rightarrow x(t) = \mathcal{L}^{-1}[(sI - A)^{-1}]x(0)$$

$$= e^{At}x(0) \quad \text{, (known before)}$$

$$\Rightarrow \boxed{\mathcal{L}^{-1}(sI - A)^{-1} = e^{At}}$$

Also, $\phi(t) = e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] = \mathcal{L}^{-1}\phi(s)$

$$\Rightarrow \phi(s) = (sI - A)^{-1} : \text{Resolvent matrix}$$

Consider forced sys:-

$$\dot{x} = Ax + Bu$$

LT

$$\Rightarrow sX(s) - x_0 = AX(s) + BU(s) \quad ; x_0 = x(0)$$

$$\Rightarrow (sI - A)X(s) = x_0 + BU(s)$$

$$\Rightarrow X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$$

$\mathcal{L}^{-1}$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]x_0 + \mathcal{L}^{-1}(sI - A)^{-1} \cdot BU(s)$$

eg Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ as sys.

Then,

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

Now

$$|sI - A| = (s-1)^2 .$$

$$\& \quad \phi(s) = (sI-A)^{-1} = \underbrace{\frac{1}{(s-1)^2}}_{\frac{1}{\det(sI-A)}} \underbrace{\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}}_{adj(sI-A)}$$

So, $\phi(s) = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$

$\mathcal{L}^{-1}$

$$\Rightarrow \phi(t) = \begin{bmatrix} e^{t} & 0 \\ te^{t} & e^{t} \end{bmatrix} = e^{At} = \text{State trans}^n \text{ matrix}$$

Aliter :- Using homogeneous eq^n sol^n method.

eg 2 Obtain time response of :-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u .$$

Given, $x^T(0) = [1 \;\; 0]$

We have :-

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We know,

$$\phi(t) = e^{At} = I + At + \frac{1}{2!}A^2t^2 + \dots$$

Substituting these values in $\phi(t)$

$$\Rightarrow e^{At} = \begin{bmatrix} 1+t+0.5t^2+\dots & 0 \\ t+t^2+\dots & 1+t+0.5t^2+\dots \end{bmatrix}$$

e^t =

$$= \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

$\to e^t$

$\to te^t$

After comput^n,

$$e^{At} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

This method is complex if matrix A is not simple. So, LT method is preferred.

Complete sol^n-

$$x(t) = \phi(t)\left[x_0 + \int_0^t \phi(-t) B u \, d\tau\right],$$

with $u = 1$.

$$\Rightarrow \phi(-\tau)Bu = \begin{bmatrix} e^{-\tau} & 0 \\ -\tau e^{-\tau} & e^{-\tau} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-\tau} \\ e^{-\tau}(1-\tau) \end{bmatrix}$$

in companion form

eg Consider a control sys. with state model

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 2\end{bmatrix}[u] \; ; \; \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}u$$

Char. eq^n :- $|\lambda I - A| = 0.$

$\Rightarrow \quad \lambda^2 + 3\lambda + 2 = 0$

eigenvalues are $\lambda_1 = -1, \lambda_2 = -2$.

Finding (M). (By Vander Monde matrix) used when sys. dynamics is in companion form

$$M = \begin{bmatrix}1 & 1\\ \lambda_1 & \lambda_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}$$

Now

$$\Lambda = M^{-1}AM = \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}.$$

$$= \begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}$$

Transformed modal matrix

$$\dot{v} = M^{-1}AMv + M^{-1}bu \; ; \; x = MV.$$

$$= \Lambda v + M^{-1}bu.$$

State trans^n matrix

$$e^{\Lambda t} = \begin{bmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{bmatrix} \qquad (\equiv e^{At})$$

Now,

$$\underbrace{v(t)}_{\text{any State}} = \underbrace{e^{\Lambda t}}_{\text{Trans}^n\text{ matrix}} \underbrace{v(0)}_{\text{Initial cond}^n}$$

} represent^n of any state, in general.

Transforming back

$\Rightarrow M^{-1} x(t) = e^{At} M^{-1} x(0)$

$\Rightarrow x(t) = e^{At} x(0)$

* If $\exists$ 3 state variables, disp, vel. & accl^n
  x_1 x_2 x_3,

Then, in state space model, what form of represent^n will be seen?

Ans :- Cannonical form

(Seen when $x_2 = \dot{x}_1$
$x_3 = \dot{x}_2$ - - -)

* Consider the analysis of o/p (i.e, we want to control o/p)

i/p $\longrightarrow$ STATES $\longrightarrow$ o/p.

To control o/p, we should have control over states. To control states, we should be able to identify these states & measure them.

OBSERVABILITY:

A sys. is completely observable if every state $x(t_0)$ can be completely identified by measurement of $y(t)$ over a finite time interval.

* CONTROLLABILITY

Dependence of state(v) on sys. i/p

M1

Gilbert Method

given :- LTI sys :-

$$\dot{x} = Ax + Bu \quad ; \; 0 < t < t_f$$

Transforming :- $\dot{v} = \Lambda v + \tilde{B} u$

$$\begin{bmatrix} \dot{v}_1 \\ \vdots \\ \vdots \\ \dot{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{bmatrix} u .$$

eq^n :- $\dot{v}_i = \lambda_i v_i + \tilde{b}_i u \quad , \; i = 1, 2, \ldots, n$

Sol^n for eq^n :-

$$v_i(t) = \underbrace{e^{\lambda_i t} v_i(0)}_{\text{due to initial cond}^n} + \underbrace{e^{\lambda_i t} \int_0^t e^{-\lambda_i t} \tilde{b}_i u(\tau) d\tau}_{\text{due to i/p}}$$

Note :-

i/p influence the states. For the states to be completely observable, every state should be defined. Now, $\because \tilde{B}$ defines i/p & controls state, it should be NON-ZERO.

Now,

$$\frac{v_i(t_f) - e^{\lambda_i t_f} v_i(0)}{e^{\lambda_i t_f}} = \int_0^t e^{-\lambda_i t} \tilde{b}_i u(\tau) d\tau$$

So, $\tilde{B} = \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \cdots & \tilde{b}_{1m} \\ \tilde{b}_{21} & & & \\ \vdots & & & \\ \tilde{b}_{n1} & & \cdots & \tilde{b}_{nm} \end{bmatrix}$

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(M2) $\dot{x} = Ax + Bu$.

↳ Its controllable iff rank of composite matrix

$$Q_c = \left[B \,\vdots\, AB \,\vdots\, \ldots \,\vdots\, A^{n-1}B \right]$$

↳ Rank = n

eg Check for controllability

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}}_{\text{companion form}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

$|\lambda I - A| = 0$ or $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -6 & -11 & -6-\lambda \end{vmatrix} = 0.$$

eigenvalues :- $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$

By Vander Monde matrix

$$M = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

Now, finding $\tilde{B}$ matrix ($= M^{-1}B$) because it controls state

If $\tilde{B} \neq 0$, controllable ✓

$$M^{-1}B = \begin{bmatrix} 3 & 5/2 & 1/2 \\ -3 & -4 & -1 \\ 1 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix}}_{\tilde{B}}$$

$\tilde{B} \neq 0$,

$\Rightarrow$ all states are controllable ✓

In canonical form, we have :-

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix} u$$

eg Using Kalman Method, check if sys is controllable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u$$

$$AB = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ 36 & 60 & 25 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 1 \\ -6 \\ 25 \end{bmatrix}$$

$$Q_c = [B \; AB \; A^2B]$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix}$$

$|Q_c| = 1((-1)) = -1 \neq 0$

So, rank = 3

i.e, sys. is completely controllable

Idea on rank :-

Take determinant. If det. $\neq 0$, the order of matrix = rank.

If det = 0, reduce one row & column & again find det. If det. $\neq 0$, now, new order = rank & so on.

Observability :- We are looking at how the system looks like that is seen with o/p matrix, i.e C matrix. So, for sys. to be observable, matrix $C \neq 0$.

eg Given LTI DE

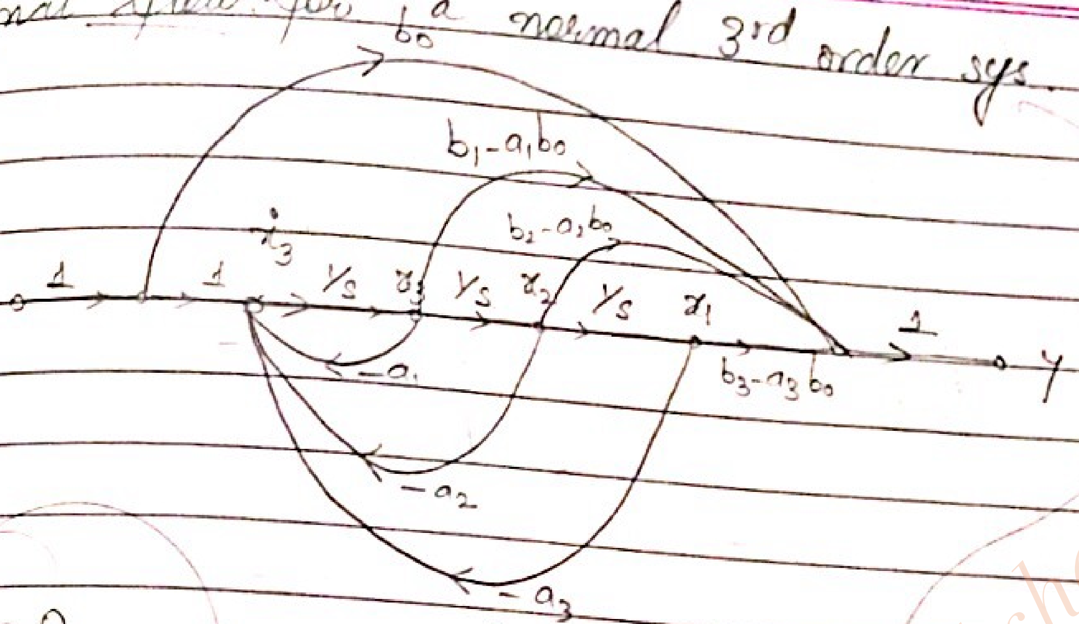
$$\ddot{y} + 2\dot{y} + y = \dot{u} + u$$

M1 Taking LT. & solving, we get

$$\frac{Y(s)}{U(s)} = \frac{s+1}{s^2+2s+1}$$

$x_1 = y$, $x_2 = \dot{y} - u$.

Signal flow for a normal 3rd order sys.



$b_0 = 0$
$b_1 = 1$
$b_2 = 1$
$a_1 = 2$
$a_2 = 1$ → use the above graph & apply the changes to make reqd graph.

Notation used:

$$TF = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

M2 State phase variable representn is:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} (b_3 - a_3 b_0) & (b_2 - a_2 b_0) & (b_1 - a_1 b_0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 u.$$

for our sys:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$$

☆ For ANY sys, no. of ENERGY storage elements (or dynamic elements) equals the ORDER of sys.

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Applying Kalman's Test

$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}; \quad AB = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$Q_c = [B \;\; AB] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$|Q_c| = 0$ So, rank $\neq 2$

Hence, rank = 1

So, out of 2 states, only 1 is controllable.

So, we dont have direct access to one of the states

Checking for Observability

The transformed canonical form representⁿ of a single o/p LTI sys is

$$\dot{v} = \underbrace{\Lambda}_{\to M^{-1}AM} v + \underbrace{\tilde{B}}_{\to M^{-1}B} u \; ; \; y = \underbrace{\tilde{C}}_{\to MC} v$$

$$= \tilde{c}_1 v_1 + \tilde{c}_2 v_2 + \ldots$$

In canonical form each state is decoupled.

Now, $v_1, v_2, \ldots$ define the observability of states.

So, if any of them = 0, not observable.

So, for observability through o/p, its corresponding coeff should be non zero.

$$\therefore \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \cdots & \tilde{C}_{1n} \\ \tilde{C}_{21} & \tilde{C}_{22} & \cdots & \tilde{C}_{2n} \\ \vdots & & & \vdots \\ \tilde{C}_{p1} & \tilde{C}_{p2} & \cdots & \tilde{C}_{pn} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$

Now, $y_i = (\quad) v_i$

→ when $\tilde{C}_{ij}$ gets multiplied with v_i

for MIMO system ← Now, if ALL elements of any ROW of C matrix = 0, then, sys is NON OBSERVABLE

→ Gilbert's Test for Observability.

* Kalman's Test for Observability.

$$Q_o = [C^T \mid A^T C^T \mid \cdots \mid (A^T)^{n-1} C^T]$$

→ rank (Q_o) should be n

* Duality Property (from Kalman's Test)

1. If pair (A, B) is controllable ⇒ $(A^T B^T)$ is observable.
2. If pair (A, C) is observable ⇒ $(A^T C^T)$ is controllable.

→ Thus, controllability & observability are dual concepts

$$\therefore \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \cdots & \tilde{C}_{1n} \\ \tilde{C}_{21} & \tilde{C}_{22} & \cdots & \tilde{C}_{2n} \\ \vdots & & & \vdots \\ \tilde{C}_{p1} & \tilde{C}_{p2} & \cdots & \tilde{C}_{pn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Now, $y_i = (\quad) v_i$

when $\tilde{C}_{ij}$ gets multiplied with v_i

for MIMO system

Now, if ALL elements of any ROW of C matrix = 0, then, sys is NON OBSERVABLE

→ Gilbert's Test for Observability.

* Kalman's Test for Observability.

$$Q_0 = \left[C^T \;\vdots\; A^T C^T \cdots \;\vdots\; (A^T)^{n-1} C^T \right]$$

→ rank (r) should be n

* Duality Property (from Kalman's Test)

1. If pair (A, B) is controllable ⇒ $(A^T B^T)$ is observable.
2. If pair (A, C) is observable ⇒ $(A^T C^T)$ is controllable.

→ Thus, controllability & observability are dual concepts

★ Phase variable form for observability.

$\dot{v} = A_0 v + B_0 u$

$y = C_0 v + du$

Indicates -ve feedback

$$A_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & & 1 & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & & 1 & -a_1 \end{bmatrix} \quad B_0 = \begin{bmatrix} B_n \\ B_{n-1} \\ B_{n-2} \\ \vdots \\ B_1 \end{bmatrix}$$

have to be derived from b values

↳ $a_1, a_2 \ldots$ are coeff of $s^{n-1}, s^{n-2} \ldots$ in the denominator of TF.

$$C_0 = [0 \; 0 \; 0 \ldots 0 \; 1]$$

Q Check for observability, given state model of a control syst.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \equiv Ax + Bu.$$

$$y = [3 \; 4 \; 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \equiv Cx$$

(M1) Gilbert method : Checking if CM has any row = 0 .

finding eigenvalues, $|\lambda I - A| = 0$

$\Rightarrow (\lambda_1, \lambda_2, \lambda_3) = (0, -1, -2)$

Now, A is in companion form

$\Rightarrow M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$; $x = MV$; $y = (CM)V$

$y = \begin{bmatrix} 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$

$\hookrightarrow$ here, V_2 is not observable

M2) Kalman's Test

$A^T C^T = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$; $A_T^2 C_T = (A_T)[(A_T) C_T] = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$

$Q_o = [C^T \,|\, A^T C^T \,|\, (A^T)^2 C^T]$

$Q_o = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$

$\hookrightarrow$ finding rank, $|Q_o| = 0$, so, rank $\neq 3$.

then, $\begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} \neq 0$, so, rank $= 2$

hence, out of 3 states, 2 states are observable

So, not completely observable.

✯ CONTROLLING THE SYSTEM

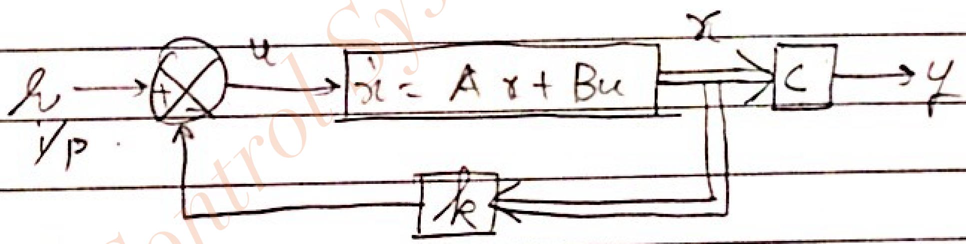
(after understanding state space models, controllability & observability)

for a sys. ~~state~~ defined as :-

$$\dot{x} = Ax + Bu \; ; \; y = Cx .$$

State variable feedback a scalar form may be

$$r = kx = [k_1 \; k_2 \; k_3 \cdots k_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$



$$\underset{\text{control i/p}}{u} = -kx + \underset{\text{reference i/p}}{r}$$

State variable feedback

using same diagram for Automatic control (FLOW CONTROL — figure (A))

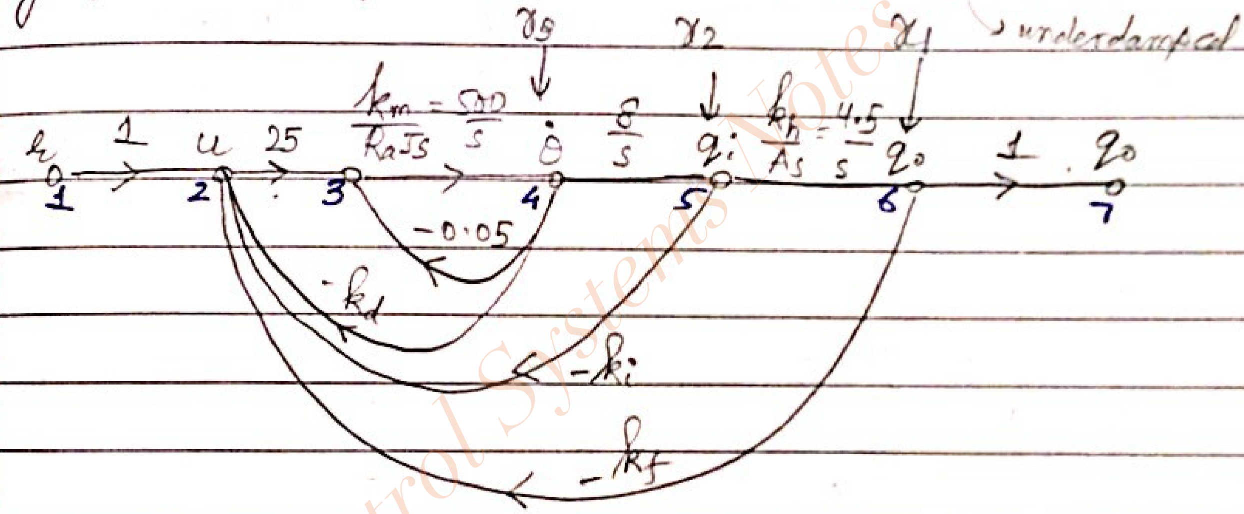
↳ what it tells ?

Automating the process of filling water in a bucket. There is a reference value r (i.e. suppose 10 cm height of water level should be filled in bucket). That is constantly checked.

ζ : damping coeff
T_s : settling time

If reference level − current level = 0, then, valve will stop.
Height of bucket is given (connected to transducer which will send the ~~dt~~ height diff. value to the difference amplifier for checking the difference & accordingly opening/closing the valve)

Q. Consider following sys. Adjust feedback sys gain for sys poles to be placed s.t it has $\zeta = 0.8$, $T_s = 0.1\,s$
↳ underdamped



Deriving gain of TF, by mason's gain formula

S1)

$$P_1 = (1234567) = (25)\left(\frac{500}{s}\right)\left(\frac{8}{s}\right)\left(\frac{4.5}{s}\right)$$

S2) Individual loops.

$$(2342) = (25)\left(\frac{500}{s}\right)(-k_d)$$

$$(343) = \left(\frac{500}{s}\right)(-0.05)$$

$$(23452) = (25)\left(\frac{500}{s}\right)\left(\frac{8}{s}\right)(-k_i)$$

$$(234562) = (25)\left(\frac{500}{s}\right)\left(\frac{8}{s}\right)\left(\frac{4.5}{s}\right)(-k_f)$$

S3)

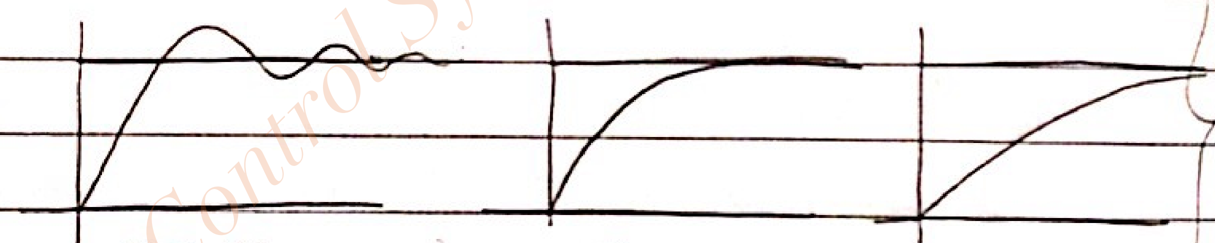
No 2 non touching

$\Delta_1 = 1$.

So, $TF = \dfrac{P_1 \Delta_1}{1 - \sum \text{individual loops}}$

$\Rightarrow TF = \dfrac{\dfrac{25 \times 500 \times 8 \times 4.5}{s^3}}{1 - \left[\dfrac{25 \times 500 \times (-k_d)}{s} + \dfrac{500 \times 0.05}{s} + \dfrac{25(500)(8)(-k_i)}{s^2} + \dfrac{25 \times 500 \times 8 \times 4.5 \times (-k_f)}{s^3}\right]}$

Note :-



$\Rightarrow TF = \dfrac{4.5 \times 10^5}{s^3 + 3.125 \times 10^5 k_d s^2 + 1 \times 10^5 k_i s + 4.5 \times 10^5 k_f} \rightarrow (1)$

Char. eq^n can be factorised as:-

$(s^2 + 2\zeta\omega_n s + \omega_n^2)(s + \zeta\omega_n) \rightarrow (2)$

* we have a modelled control sys. & an external power/controller that we put
We are modifying the poles of the controller

Now, $T_s = \frac{\ln(2\%)}{\zeta \omega_n}$ (assuming 2% tolerance band)

$\Rightarrow 0.1 = \frac{4}{\zeta \omega_n}$

$\Rightarrow \omega_n = \frac{4}{(0.1)(0.8)}$

$= \frac{\cancel{100} \times 4}{\cancel{8}}$

$\Rightarrow \omega_n = 50.$

Substituting in eq^n (2)

$\Rightarrow (s^2 + 2 \times 0.8 \times 50 s + 2500)(s + 0.8 \times 250)$

$\Rightarrow s^3 + 120 s^2 + 5700 s + 10^5 \rightarrow (3)$

Comparing eq^n (1) & (3).

$\Rightarrow 4.5 \times 10^5 k_f = 10^5 \Rightarrow k_f = 0.222$

$(1.0 \times 10^5) k_i = 5.7 \times 10^3 \Rightarrow k_i = 0.057$

$2.5 + 12.5 \times 10^2 k_d = 120 \Rightarrow k_d = 0.75 \times 10^{-3}$

So, that's how we find f/b gains k_i, k_f, k_d.

Now, our form is $\dot{x} = Ax + Bu$.
& we had $u = -kx + r$.
Using both, we get

$$\dot{x} = (A - BK)x + Br.$$

Assume $v = Px$
$\therefore x = P^{-1} v.$

$$\dot{v} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_2 & -a_1 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u.$$

★ ζ & ω_n define the trajectory of o/p

★ t_s, t_p, t_r, M_p: defined by ζ & ω_n

$$\Rightarrow \boxed{\dot{V} = A_c V + B_c u}$$

Now, $u = -kx + r$ changes to

$$u = -k P^{-1} V + r$$

$$\boxed{u = -K_c V + r}$$

$$\boxed{K_c = K P^{-1}}$$

The sys. dynamics matrix, after transformⁿ is of A-BK

$(A_c - B_c K)$

let its eigenvalues be $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$

So, characteristic eqⁿ:

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n = 0$$

Converting given dynamics to controllable form using P matrix

feedback coeff of controllable phase var.

Assuming $K_c = [\alpha_n - a_n, \alpha_{n-1} - a_{n-1}, \ldots, \alpha_1 - a_1]$

Then, sys is transformed to

$$\dot{V} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ & & & & & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & \cdots & \cdots & -\alpha_1 \end{bmatrix} V + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} r$$

Now, $\boxed{K = K_c P}$

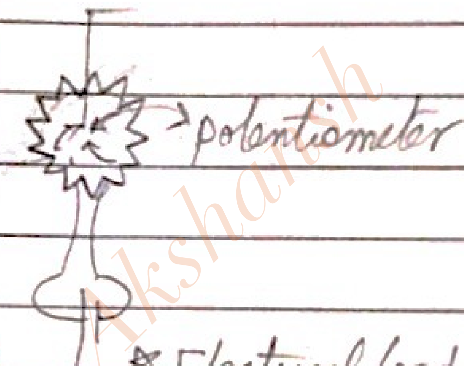
$$\Rightarrow K = [\alpha_n - a_n, \alpha_{n-1} - a_{n-1}, \ldots, \alpha_1 - a_1] P$$

☆ Pole loc^{ns} = eigen values of char. eq^n.
☆ Time constt order → Mech. sys : sec.
→ Electrical sys : mili sec

Designing State Controller → fig 12.23 (Ifeachor)

Q. Considering fig. of an electrical sys.
From the fig, we find 3 loops.

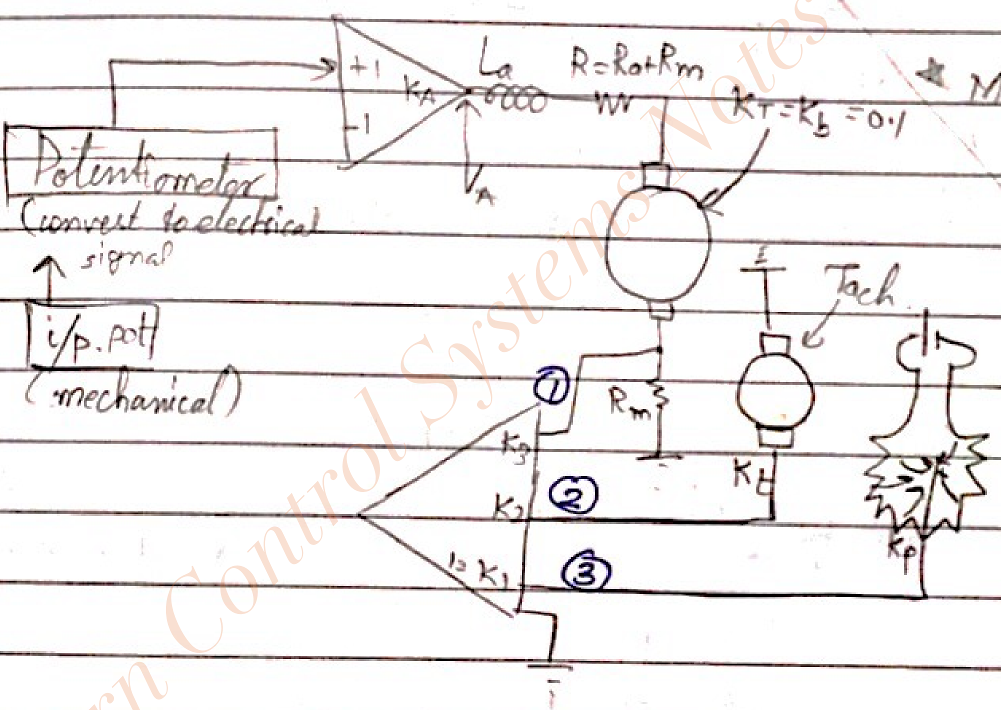
Pos^n control sys. ⇐
- 1 → motor control
- 2 → speed control
- 3 → angular pos^n control



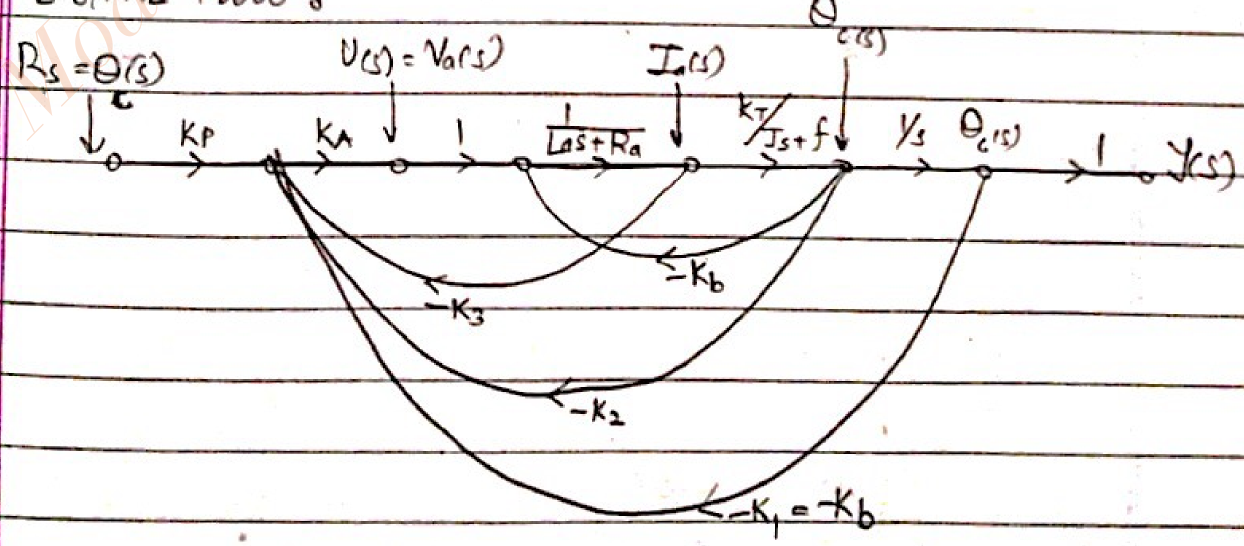
Design f/b control coeff : K_1, K_2, K_3, K_b

☆ Electrical load
⇓
current

☆ Mechanical load
⇓
Torque.

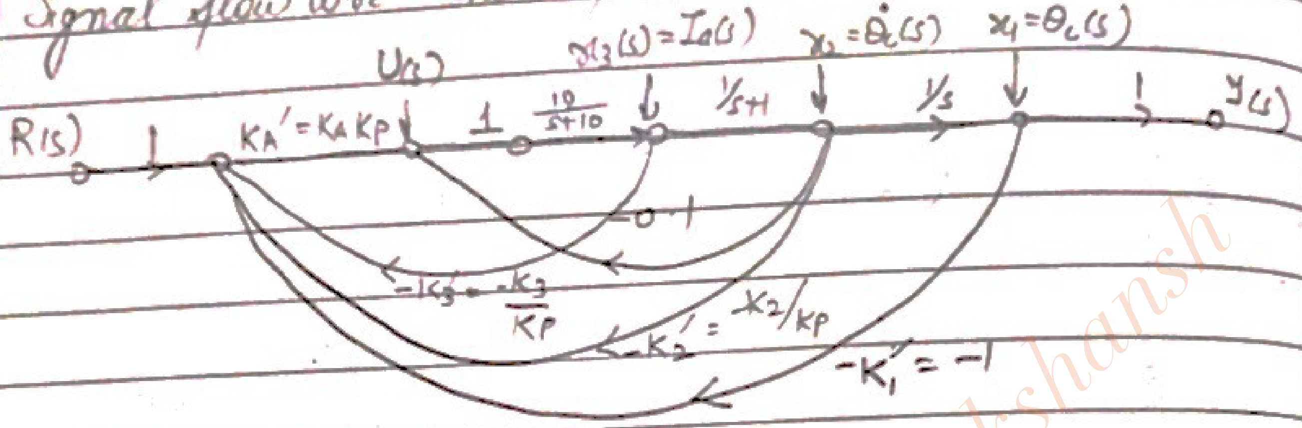


SIGNAL FLOW :



Making state space model from signal flow:

Signal flow wrt state space model.



$$\frac{x_2}{s} = x_1 \Rightarrow x_2 = s x_1 = \dot{x}_1$$

$$\Rightarrow \dot{x}_1 = x_2$$

$$\frac{x_3}{s+1} = x_2 \Rightarrow x_3 = s x_2 + x_2 = \dot{x}_2 + x_2$$

$$\Rightarrow \dot{x}_2 = x_3 - x_2 \quad \&$$

$$\frac{(-0.1 x_2 + u)10}{s+10} = x_3$$

$$\Rightarrow 10u - 10x_3 - 1x_2 = s x_3 = \dot{x}_3$$

So,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u .$$

$$= Ax + Bu$$

$$y = [1 \ 0 \ 0] x$$

Now, checking for controllability,

$$Q_c = [A \mid AB \mid A^2 B]$$

$$Q_c = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 10 & -110 \\ 10 & -100 & +990 \end{bmatrix}$$

$|Q_c| \neq 0 \Rightarrow$ its controllable.

Now, specs given to us are $\zeta = 0.5$, $\omega_n = 2$ rad/s
As per these, find out k_1, k_2, k_3, k_b

So, char. eq^n for triple pole (3rd order):

$$(s + \zeta\omega_n)(s^2 + 2\zeta\omega_n s + \omega_n^2) = 0$$

for 2 poles :- $\underbrace{s^2 + 2s + 4}$
↳ taking eq^n :- $s^2 + 2(0.5)(2)s + (2)^2$

Solving : $s_1, s_2 = -1 \pm j\sqrt{3}$.

3rd pole :- $s_3 = -\zeta\omega_n$
To ensure, we achieve our specs, making it 10 times.

$$\text{So, } s_3 \leq -10\zeta\omega_n = -10$$

Den. of CLTF :-

$$D(s) = (s+10)(s+1+j\sqrt{3})(s+1-j\sqrt{3})$$
$$= s^3 + 12s^2 + 24s + 40$$

In terms of Eigenvalues :-

$$\lambda^3 + 12\lambda^2 + 24\lambda + 40 = 0 \quad (\text{desired})$$

↳ we can get values of α = (roots of it) = (40, 24, 12)

Now, char. eq^n of $A = |\lambda I - A| = 0$

$$\Rightarrow \lambda^3 + 11\lambda^2 + 11\lambda = 0$$

$$a_3 = 0, \; a_2 = 11, \; a_1 = 11$$

Now, finding

$$K_c = [K_{c1} \; K_{c2} \; K_{c3}] = [(\alpha_3 - a_3) \; (\alpha_2 - a_2) \; (\alpha_1 - a_1)]$$
$$= [(40-0) \; 1(24-11) \; (12-11)]$$

$\Rightarrow \quad K_c = [40 \quad 13 \quad 1]$

Now

$$K = K_c P$$

$$\hookrightarrow P = \begin{bmatrix} P_1 \\ P_1 A \\ P_1 A^2 \end{bmatrix}$$

where $P_1 = [0 \quad 0 \quad 1] \, Q_c^{-1}$ → B_c matrix (Controlled format) ↳ companion form

$$= [0 \quad 0 \quad 1] \begin{bmatrix} 1.1 & 1 & 0.1 \\ 1.1 & 0.1 & 0 \\ 0.1 & 0 & 0 \end{bmatrix}$$

$$P_1 = [0.1 \quad 0 \quad 0]$$

Now,

$$P_1 A = [0 \quad 0.1 \quad 0]$$

$$P_1 A^2 = [0 \quad -0.1 \quad 0.1]$$

So,

$$P = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & -0.1 & 0.1 \end{bmatrix}$$

Now $K_c = [40 \quad 13 \quad 1]$

$\Rightarrow$

$$K = K_c P$$

$$= [4 \quad 1.2 \quad 0.1]$$

$$\equiv [k_1 \quad k_2 \quad k_3]$$

So,

$$k_1 = k_1' K_A' = 4$$

$$k_2 = k_2' K_A' = 1.2$$

$$k_3 = k_3' K_A' = 0.1$$

Now, $k_1' = 1$.

$\Rightarrow K_A' = 4$

& $k_2' = 0.3$

& $k_3' = 0.025$

This will meet the design specs

Summary :- $\boxed{\text{CONTROLLABILITY}}$

Given $\dot{x} = Ax + Bu$ & $y = Cx$.

It is converted to

$$\dot{V} = A_c V + B_c u. \rightarrow (1)$$

We found matrix P to convert $A \rightarrow A_c$ & $B \rightarrow B_c$.

where $A_c = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & & -a_1 \end{bmatrix}$, $B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

finding matrix P :-

$$V = Px \; ; \; P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & \cdots & \cdots & P_{2n} \\ \vdots & & & \vdots \\ P_{n1} & & & P_{nn} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}$$

$$P_i = [P_{i1} \;\; P_{i2} \cdots P_{in}] \; ; i = 1, 2, \cdots n.$$

i.e,

$$V(t) = Px = \begin{bmatrix} P_1 \\ P_1 A \\ \vdots \\ P_1 A^{n-1} \end{bmatrix} x.$$

& P_1 should satisfy condⁿ :-

$$P_1 B = P_1 AB = P_1 A^2 B = \cdots = P_1 A^{n-2} B = 0$$

$\hat{x}$ → signifies its estimated value

Now, for finding P, we need to find P_1 where

$$P_1 \times \underbrace{[B \mid AB \mid \dots \mid A^{n-1}B]}_{Q_c} = [0\ 0\ 0 \dots 1]$$

$$\Rightarrow \boxed{P_1 = [0\ 0 \dots 1]\, Q_c^{-1}}$$

gives

Note: $A \rightarrow A_c$ as

$$\dot{v} = PAP^{-1}v + PBu$$

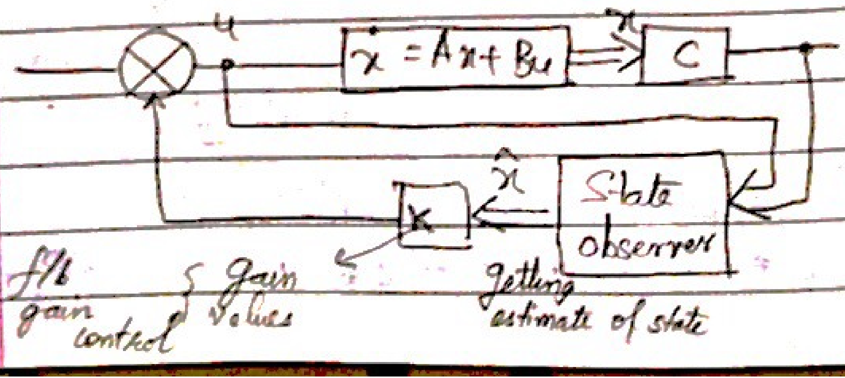
Comparing

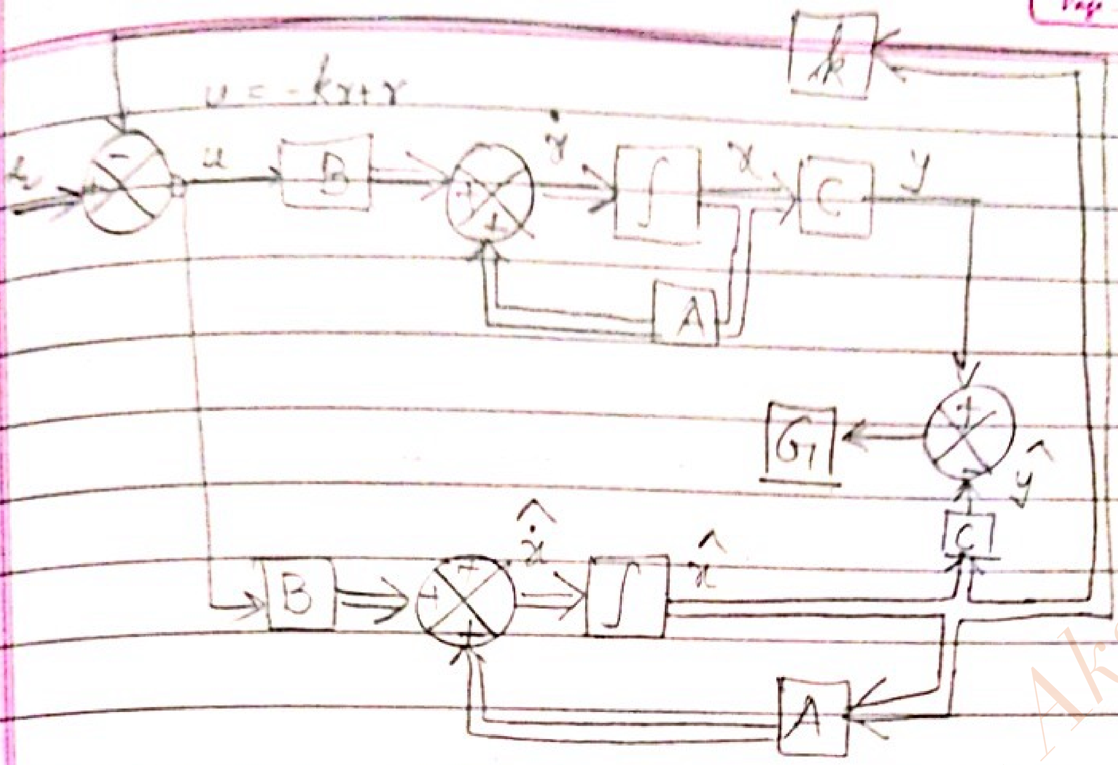
$$B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = PB = \begin{bmatrix} P_1 B \\ P_1 AB \\ \vdots \\ P_1 A^{n-1} B \end{bmatrix}$$

Now, OBSERVABILITY.

↳ modelling a physical sys with mathematical expression & seeing o/p from actual & modelled sys. If 2 o/ps are same, then, 2 states are same. So, if we can get the state in model, we say, its there in real

A linear state observer:





Idea: Design G s.t. $y - \hat{y}$ is min

Analysis:

$$\dot{\hat{x}} = A\hat{x} + G(y - \hat{y}) + Bu$$

$$y = Cx, \quad \hat{y} = C\hat{x}$$

$$\Rightarrow \dot{\hat{x}} = A\hat{x} + G(Cx - C\hat{x}) + Bu$$

Now

$$\dot{x} = Ax + Bu$$

$$\Rightarrow \dot{x} - \dot{\hat{x}} = (Ax + Bu) - \left(A\hat{x} + G(Cx - C\hat{x}) + Bu\right)$$

$$= (A - GC)x - (A - GC)\hat{x}$$

$$\Rightarrow \dot{x} - \dot{\hat{x}} = (A - GC)(x - \hat{x})$$

$$\Rightarrow \dot{\tilde{x}} = (A - GC)(\tilde{x})$$

$$\tilde{x} = x - \hat{x}, \quad \dot{\tilde{x}} = \dot{x} - \dot{\hat{x}}$$

= Diff b/w actual & estimated value

* o/p for observer = values we are estimating

To check for Observability using Gilbert or Kalman method. (pair A, C is seen)

Gilbert : find $\tilde{C} = CM$
↳ all elements of $\tilde{C} \neq 0$
⇒ observable

Kalman: find $Q_o = [C^T \; A^T C^T, \; ... \; (A^T)^{n-1} C^T]$
↳ find rank (Q_o) = same as order of sys
⇒ observable (completely)

By dual property,
if $(A\,C)$ is observable, $(A^T \, C^T)$ is also observable
Finding eigenvalues of $(A^T - C^T G^T)$
find $(A - GC)^T$ or $(A - GC)$ eigenvalues

eg Consider a sys given as

$$\dot{x} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u.$$

$(\equiv Ax + Bu)$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x.$$

$(\equiv Cx)$

Checking for observability using Kalman method

$$C^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{3\times 1} \quad A^T = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}_{3\times 3}$$

finding rank of $Q_o = \left[C^T \quad A^T C^T \ldots \right]$

we find its rank = 3, so observable.

To find: Design state observer so eigenvalues of $(A - GC)$ are at $-3 \pm j1$ → its given specification

$$(a + jb = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2})$$

→ So, ζ & ω_n can be (found by comparing)

Solⁿ let $G = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$

$$(A - GC) = \begin{bmatrix} 3 & 2 & -g_1 \\ 3 & -1 & 1-g_2 \\ 0 & 2 & -g_3 \end{bmatrix}$$

Char. eqⁿ :- $|\lambda I - (A - GC)| = 0$

$$\Rightarrow \begin{vmatrix} \lambda-1 & -2 & g_1 \\ -3 & \lambda+1 & g_2-1 \\ 0 & -2 & \lambda+g_3 \end{vmatrix} = 0$$

here, in real, ∃ only 2 pole values : $-3 \pm j1$ meaning, these 2 pole locⁿˢ should have dominant char. So, out of 3 poles, 3rd pole has to be kept far away s.t it has less influence. So, keeping it at 4.

→ gives $\lambda^3 + g_3\lambda^2 + (2g_2 - 9)\lambda + 2 + 6g_1 - 2g_2 - 7g_3 = 0$ → Modelled

Now, Desired char. eqⁿ. is

$$(\lambda + 3 + j1)(\lambda + 3 - j1)(\lambda + 4) = 0$$

$$\Rightarrow \lambda^3 + 10\lambda^2 + 34\lambda + 40 = 0 \longrightarrow \text{Actual}$$

Comparing modelled & actual to find g_1, g_2, g_3.

$$\Rightarrow g_1 = 25.2,\ g_2 = 21.5,\ g_3 = 10$$

cts : continuous

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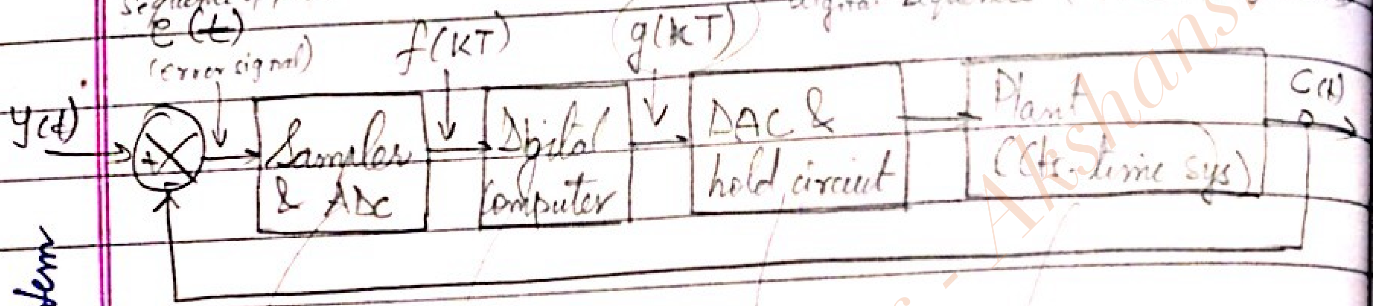
DIGITAL CONTROL SYSTEM

When a control sys. has digital component involved in it

→ o/p of digital comp. in discrete digital sequences (T: sampling interval)

sequence of poles



always cts. sys.

Sampled-data control sys or Digital control System

Sampling ⇒ making a cts signal discrete. ADC gives digital data which can be directly fed on comp.

Works on digital signals. So, any i/p coming is converted to digital using ADC. Operⁿ: control operⁿ are performed in comp (eg. PID controller will try to minimize error signal e(t))

o/p has to be made in cts. So, DAC is used. The discrete sequence is converted to analog

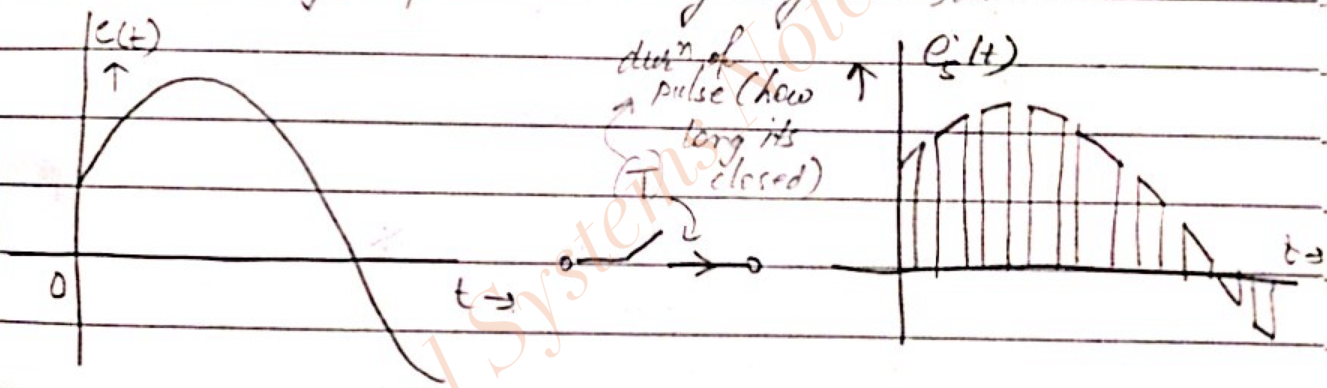
★ Application :

1. For long distance communicⁿ/transmission. (∵ digital transmⁿ better over analog) ↳ losses are less etc.
2. Pulse Amplitude Modulⁿ (PAM)
   ↳ pulses transferred to & from sys.
   eg: If c(t) is speed, that is fed back to i/p. The amount of speed is seen by digital devices (with pulse

3. Can be used in Time Division Multiplexing (TDM) mode.

Many parameters can be monitored.
eg. Consider many block diagrams — one giving speed o/p, other giving temp., other pressure etc. All these o/p are fed back to a complete sys. block diagram which uses all these data to evaluate the control sys. All this can be done in TDM mode.

So, basically, for an analog signal like:



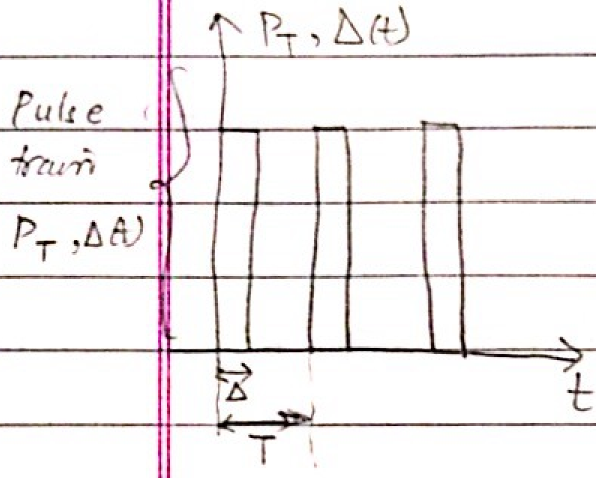
UNIFORM PERIODIC SAMPLING

* Sampling

Multiorder	Multirate	Random Sampling
Periodic Repetition of sampling pattern	2 simultaneous sampling option at diffᵗ time periods to produce sampled o/p	Sampling instances are random

☆ Spectrum Analysis of sampling process

Consider unit impulse train of signals:



approxim^n of signal :–

$$e_s(t) = e(t) . \Delta \delta_T(t)$$

unit impulse train

$$\delta_T(t) = \sum_{k \to -\infty}^{\infty} \delta(t - kT)$$

cts analog signal

So, $e_s(t) = \Delta e(t) \sum_{k \to -\infty}^{\infty} \delta(t - kT)$

Applying FT (to see in freq domain)

$$\Rightarrow \mathcal{F}(e_s(t)) = \frac{\Delta}{2\pi} \mathcal{F}(e(t)) * \mathcal{F}\left[\sum_{k \to -\infty}^{\infty} \delta(t - kT)\right]$$

$$\because \mathcal{F}(f(t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Now,

$$\mathcal{F}\left[\sum_{k \to -\infty}^{\infty} \delta(t - kT)\right] = \omega_s \sum_{k \to -\infty}^{\infty} \delta(\omega - k\omega_s)$$

$\because \exists$ train of impulses

$\omega_s = \frac{2\pi}{T}$ = Sampling freq.

So, we have

$$E_s(\omega) = \frac{\Delta}{2\pi} E(\omega) * \omega_s \sum_{k \to -\infty}^{\infty} \delta(\omega - k\omega_s)$$

Now,

$$E(\omega) * \delta(\omega - \omega_0) = E(\omega - \omega_0)$$

$$\Rightarrow E_s(\omega) =$$

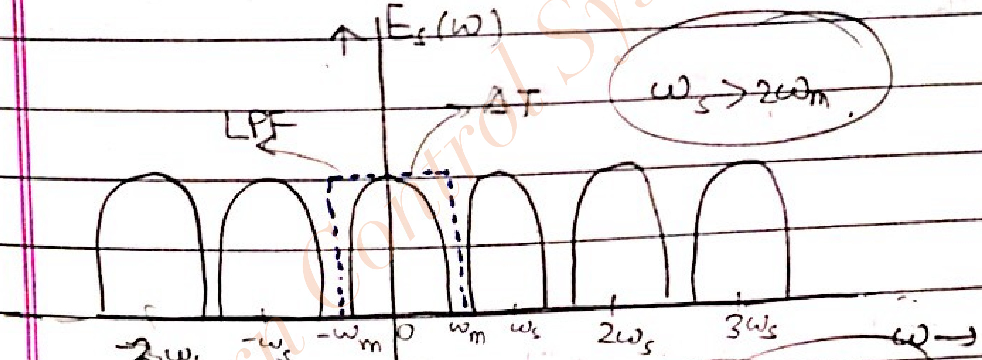
\#

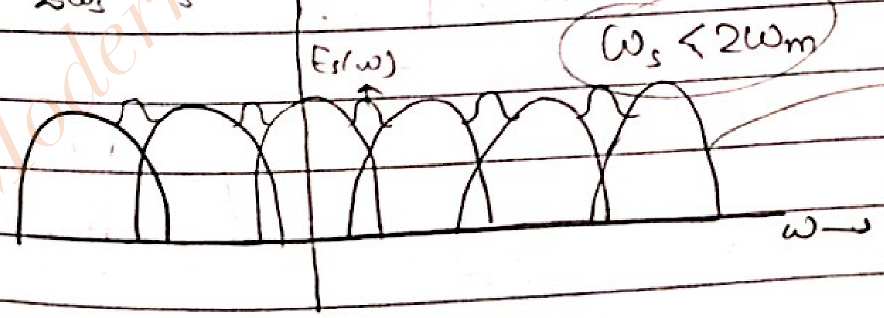
* Sampling freq. s.t info is retained & Any signal freq. of i/p signal ≤ sampling freq. This is called

SHARON'S SAMPLING THOREM

Sampling freq in real = 4 × (i/p freq - max freq content)

If we choose





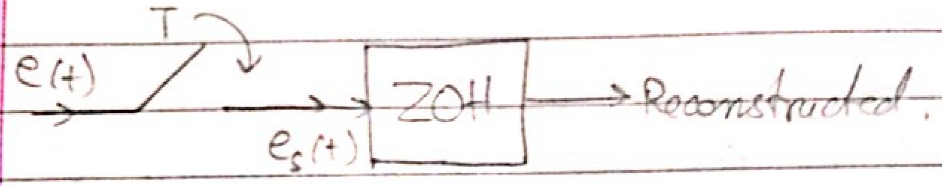
→ overlap is seen when $\omega_s < 2\omega_m$

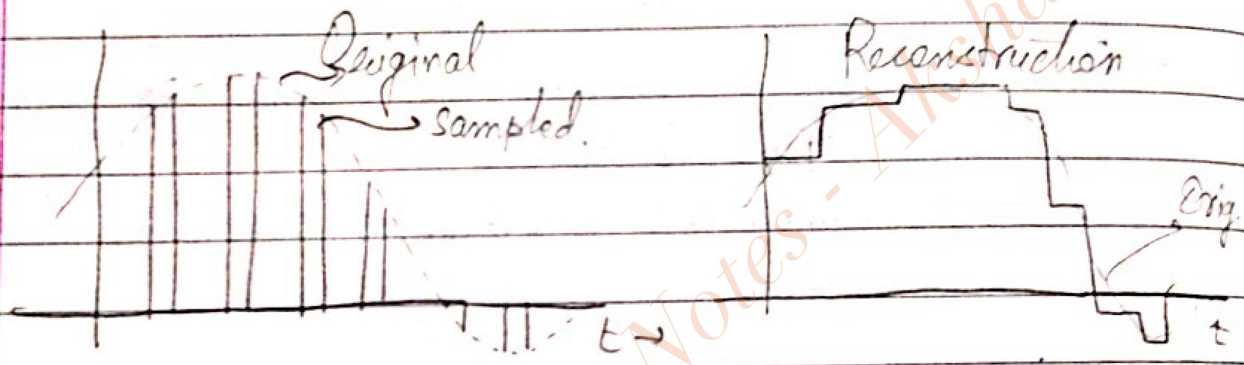
This is called ALIASING.

↳ DSP, Ch-2

ZOH : Zero Order Hold.
o/p is held just like few order value

☆ Sampler & zero order hold





Difference eq^{ns} :- Consider 1st ord. CTBS^n

$$C(t) = A\int_0^t [r(t) - C\tau]\, d\tau$$

or

$$c(t) + A\int_0^t c(\tau)\,d\tau = A\int_0^t r(\tau)\,d\tau$$

Difference eq^n:-

$$c(kT) + A\underbrace{\int_0^{kT} c(\tau)\,d\tau}_{SSS} = A\underbrace{\int_0^{kT} r(\tau)\,d\tau}_{SSS\ approxim^n}.$$

So,

$$\underbrace{c(kT)}_{\text{present o/p}} + A\underbrace{\sum_{m=0}^{k-1} c(mT)T}_{\text{prev. O/Ps}} = A\underbrace{\sum_{m=0}^{k-1} r(mT)T}_{\text{prev. i/ps}} \quad (1)$$

$(\tau = mT)$

so that o/p can be anticipated

Now, next instant :-

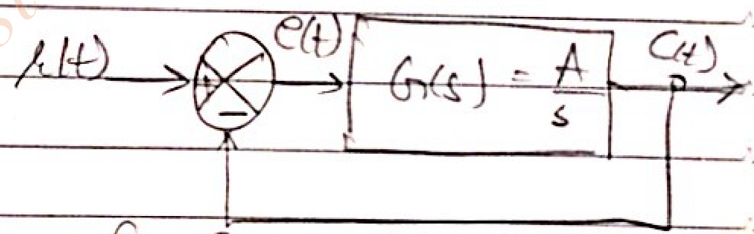
till present

$$c(kT+T) + A\sum_{m=0}^{k} c(mT)T = A\sum_{m=0}^{k} r(mT)T \quad \text{--- (2)}$$

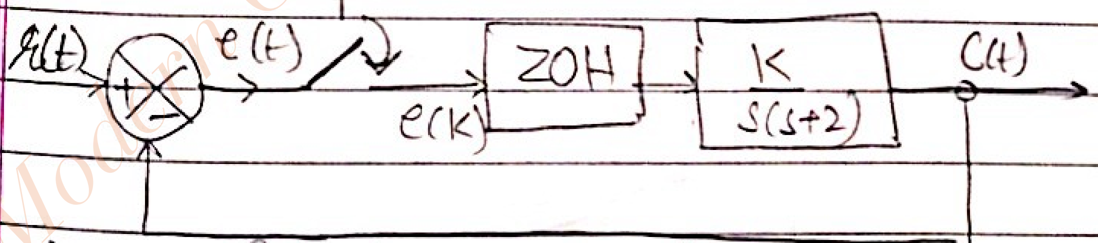
Subtracting (1) & (2)

$$\Rightarrow c[(k+1)T] - c(kT) + \underbrace{ATc(kT)}_{\left(A\sum_{m=0}^{k} c(mT)T - A\sum_{m=0}^{k-1} c(mT)T\right)} = ATr(kT)$$

or, $\underbrace{c(k+1)}_{\text{next Sample estimation}} = (1-AT)\,c(k) + AT\,r(k)$



If sampler & holder is also there :



During k^{th} time interval

$$e_k(t) = e(kT) \;;\; kT \leq t < (k+1)T$$

Now,

$$c(t) = c(kT) + Ae(kT)(t-kT);\; kT \leq t < (k+1)T$$

$$c[(k+1)T] = c(kT) + ATe(kT)$$

So,
$$c(k+1) = c(k) + AT\,e(k)$$
$$e(kT) = r(kT) - c(kT)$$

which can be written as:-

$$\boxed{c(k+1) = (1-AT)\,c(k) + AT\,r(k)}$$

$$c(1) = (1-AT)\,c(0) + AT\,r(0)$$
$$c(2) = (1-AT)\,c(1) + AT\,r(1)$$

found

In general:-

$$c(k) = (1-AT)^k\,c(0) + \sum_{i=0}^{k-1} (1-AT)^{k-1-i}\,r(i)$$

o/p at t=0
(initial o/p)

So, response is due to zero i/p + reference i/p.

forced i/p,

CHAPTER CONCEPT

$\boxed{\text{Z-transform}}$:-

It is discrete signal Fourier transform.
Gives freq. info of sampled signals.

One sided z-transform of a sequence

$$f(k) = \{0, 0 \ldots f(0), f(1), f(2) \ldots\}$$

is defined as:-

Way of thinking: Sys. is diverging ⇒ seq. → ∞, if seen as o/p sequence, then, sys. is unstable

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$$Z[f(k)] = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$

↳ $k \geq 0$ (+ve sequence)

eg ① Consider a GP :-

$$f(k) = \{ \ldots 0, 1, a, a^2, \ldots \}$$

Write this sequence in power series ≡

$$f(k) = \begin{cases} a^k & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$$

↳ a : real no.

Now, z-transform of this sequence,

$$Z[f(k)] = F(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} (az^{-1})^k$$

(Note :- It's a +ve sequence power series. So, it can be written as $\frac{1}{1-x}$ $(= 1 + x + x^2 + \ldots)$)

This infinite sys should CONVERGE.

iff :- $|(az^{-1})| < 1$ $\left(\therefore F(z) = \frac{1}{1-az^{-1}} \right)$

or $|z| > |a|$

&, the sys. DIVERGES (unbounded), if $|z| < |a|$ (i.e., $|az^{-1}| > 1$)

In the region of convergence,

$$Z[a^k] = \sum_{k=0}^{\infty} (az^{-1})^k = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

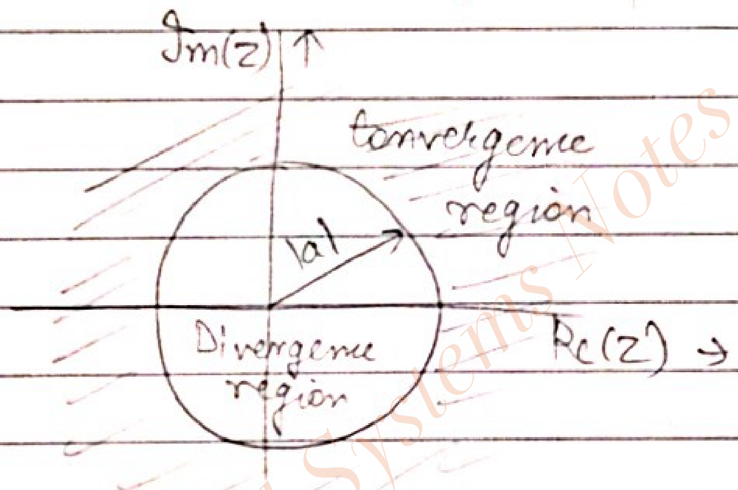
Note: Its in +ve power of Z.

So, we can write

$$Z[a^k] = F(z) = \begin{cases} \frac{z}{z-a} & ; |z| > |a| \\ \text{unbounded} & ; |z| < |a| \end{cases}$$

→ We have diff^t freq components available in discrete time signals

→ conclusion for series a^k



eg ② Consider

$$f(k) = \begin{cases} \sin k\omega T & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$$

$$Z[\sin k\omega T] = F(z) = \sum_{k=0}^{\infty} (\sin k\omega T) z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^{jk\omega T} - e^{-jk\omega T}}{2j} \right) z^{-k}$$

$$= \underbrace{\sum_{k=0}^{\infty} \frac{e^{jk\omega T} z^{-k}}{2j}}_{a} - \underbrace{\sum_{k=0}^{\infty} \frac{e^{-jk\omega T} z^{-k}}{2j}}_{b}$$

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$$a = \sum_{k=0}^{\infty} \frac{e^{jk\omega T} z^{-k}}{2j} = \frac{1}{2j} \sum_{k=0}^{\infty} \left(e^{j\omega T} z^{-1}\right)^{k}$$

$$= \frac{1}{2j} \frac{z}{z - e^{j\omega T}} \; ; \; \underbrace{|z|>1}$$

$a=1$
So, $|z|>|a|$

||ly, $b = \frac{1}{2j}\left(\frac{z}{z - e^{-j\omega T}}\right)$

So,

$$Z[\sin k\omega T] = \frac{z \sin \omega T}{\{z - e^{j\omega T}\}\{z - e^{-j\omega T}\}}$$

$$\Rightarrow Z[\sin k\omega T] = \frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1} \; ; \; \underbrace{|z| > 1}_{\text{cond}^n \text{ for convergence}}$$

$>|a|$

Z transform lies outside unit circle $\Rightarrow$ converging

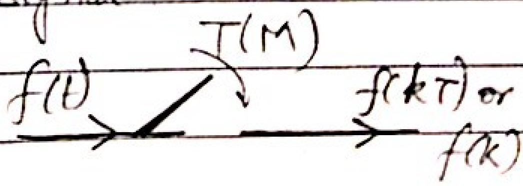
Suppose region of convergence is defined as

$|z| \geq R \rightarrow$ radius of sequence.

Now,

Cts time $\longrightarrow$ sampled signal

$f_s = f(kT) \rightarrow$ sampled.



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eg 3) Now, consider a time fn,

$$f(t) = e^{j\omega t}\, u(t)$$ making value exist only in +ve region

Now,

$$\underbrace{f(k)}_{f(t)\big|_{t=kT}} = e^{j\omega(kT)}\, u(k) = \{e^{j\omega T}\}^{k} ; k \geq 0.$$

Doing Z transform ; $a = e^{j\omega T}$

$$Z\left[e^{j\omega kT}\right] = \frac{z}{z - e^{j\omega T}} \quad ; \quad |z| > 1$$

eg 4) Consider

$$f(t) = \cos\omega t$$

so, $f(k) = \cos k\omega T$

$$Z[\cos\omega kT] = Z\left[\frac{e^{j\omega kT} + e^{-j\omega kT}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right]$$

$$= \frac{z(z - \cos\omega T)}{z^2 - 2z\cos\omega T + 1} \quad ; \quad |z| \geq 1$$

★ Reconstructing discrete time domain signal
(inverse Z transform)

We found $f(k) = Z^{-1}[F(z)]$

we know $Z[a^k] = \frac{z}{z-a}$

Z & Z^{-1} are duals (assumed)

$\Rightarrow Z^{-1}\left[\frac{z}{z-a}\right] = a^k$. Pair: $\boxed{\frac{z}{z-a} \Leftrightarrow a^k}$ ★

IIly, $Z^{-1}\left[\frac{z \sin \omega T}{z^2 - 2z\cos\omega T + 1}\right] = \sin k\omega T$

Pair: $\boxed{\frac{z \sin \omega T}{z^2 - 2z\cos\omega T+1} \Leftrightarrow \sin k\omega T}$ ★

So, the above are transformed pairs
Shortcut method to derive discrete time domain signal

Finding more pairs:-

we have :- $\sum_{k=0}^{\infty} a^k z^{-k} = \frac{z}{z-a}$

Differentiating both sides wrt z.

$\Rightarrow [0 - az^{-2} - 2a^2 z^{-3} \ldots] = -\frac{a}{(z-a)^2}$.

$\left.\frac{d}{dz}\right|_{k=1}$ $\times$ both sides by $(-z^2)$

So, $[a + 2a^2 z^{-1} + \ldots] = \frac{az^2}{(z-a)^2}$.

$$\left[\sum_{k=1}^{\infty} k a^k z^{-(k-1)}\right] = \frac{az^2}{(z-a)^2}$$

$k=1 \rightarrow k=0$ as starting term

$k \rightarrow k+1$

$$\Rightarrow \sum_{k=0}^{\infty} (k+1) a^{k+1} z^{-k} = \frac{az^2}{(z-a)^2}.$$

So, another pair :-

$$\boxed{(k+1) a^{k+1} \longleftrightarrow \frac{az^2}{(z-a)^2}}$$

Another commonly used signal :- (finding pair)

$$\delta(k) = \{1, 0, 0, \ldots\} = 1 \; ; k=0$$
$$= 0 \; ; k \neq 0$$

Now,

$$Z[\delta(k)] = \sum_{k=0}^{\infty} \delta(k) z^{-k}.$$

$\hookrightarrow$ value exists only at $k=0$.

So, $Z[\delta(k)] = 1$.

So, $\boxed{\delta(k) \longleftrightarrow 1}$

IIIly, another pair (unit step f^n),

$$u(k) = \{1, 1, \ldots 1\} = 1, \; k \geq 0$$
$$0, \; k < 0.$$

$$Z[u(k)] = Z[1^k]$$
$$\Rightarrow a^k|_{a=1}$$

So, $\boxed{u(k) \longleftrightarrow \frac{z}{z-1}}$

PROPERTIES

LINEARITY Property

P1) $$Z[a f(k) + b g(k)] = \sum_{k=0}^{\infty} [a f(k) + b g(k)] z^{-k}$$

$$= a \sum_{k=0}^{\infty} f(k) z^{-k} + b \sum_{k=0}^{\infty} g(k) z^{-k}$$

$$= a F(z) + b G(z)$$

↳ Obeys POS & homogeneity property

P2) Shifting Property

(a) when shifted to left (advanced),

$$g(k) = f(k+1) \; ; \; k \geq -1$$

Now,

$$Z[f(k+1)] = \sum_{k=0}^{\infty} f(k+1) z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k+1) z^{-k} \cdot \frac{z}{z}$$

$$= z \sum_{k=0}^{\infty} f(k+1) z^{-(k+1)}$$ → to make argument same

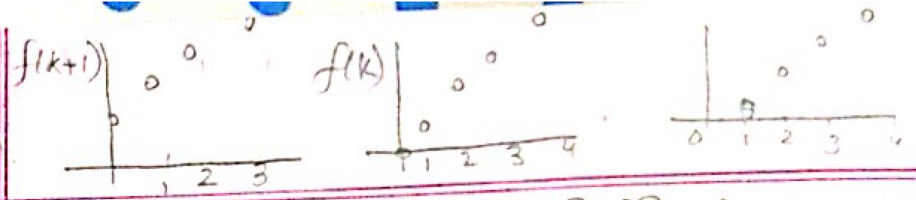
$$\Rightarrow Z[f(k+1)] = z F(k+1)$$

let $(k+1) = m$

$$\Rightarrow Z[f(k+1)] = z \sum_{m=1}^{\infty} f(m) z^{-m}$$

$$= z \left[\sum_{m=1}^{\infty} f(m) z^{-m} + f(0) z^{-0} - f(0) z^{-0} \right]$$

Sample Sequence represent



$$\Rightarrow Z[f(k+1)] = Z\left[\sum_{m=0}^{\infty} f(m)z^{-m} - f(0)\right]$$

$$= zF(z) - zf(0)$$

In general,

$$Z[f(k+n)] = z^n F(z) - \sum_{i=0}^{n-1} f(i)\, z^{n-i} \; ; k \geq -n.$$

$\rightarrow Z[f(k)]$

(b) When shifting to right,

$$g(k) = f(k-n) \; ; \; k \geq n.$$

$$Z[f(k-n)] = \sum_{k=0}^{\infty} f(k-n) z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k-n) z^{-k} \times \left(\frac{z^{-n}}{z^{-n}}\right)$$

$$= z^{-n} \sum_{k=0}^{\infty} f(k-n) z^{-(k-n)}$$

let $k-n = m$

So, $k \to 0$ to ∞

Hence, $m \to -n$ to ∞.

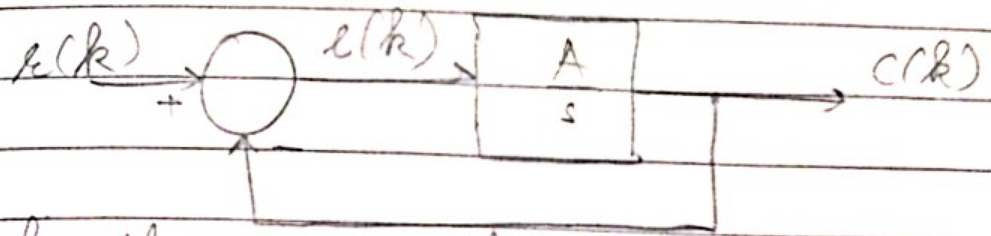
So, $Z[f(k-n)] = Z[f(m)]$

$$= z^{-n} \sum_{m=-n}^{\infty} f(m) z^{-m}$$

As $f(m) = 0$ for $m < 0$

$$\Rightarrow Z[f(k-n)] = z^{-n} \sum_{m=0}^{\infty} f(m) z^{-m} = z^{-n} F(z).$$

Now, for the sys :



for this sys, we found.

$$c(k+1) - (1-AT)\,c(k) = AT\, r(k)$$

Taking Z transform on both sides.

$$\Rightarrow Z[c(k+1)] - (1-AT)Z[c(k)] = AT\,Z[r(k)]$$

$$\Rightarrow z[C(z) - c(0)] - (1-AT)C(z) = AT\,R(z)$$

$$\Rightarrow C(z)[z - (1-AT)] = zc(0) + AT\,R(z)$$

$$\Rightarrow \underset{\text{o/p}}{C(z)} = \underbrace{\left(\frac{z}{z-(1-AT)}\right)}_{\substack{\text{due to initial} \\ \text{cond}^n \\ \text{(zero i/p - o/p)}}} c(0) + \underbrace{\left(\frac{AT}{z-(1-AT)}\right)}_{\substack{\text{due to} \\ \text{reference i/p.} \\ \text{(forced-o/p)}}} R(z)$$

For TF, we assume initial cond^n = 0.

So,

$$\frac{C(z)}{R(z)} = \frac{AT}{z-(1-AT)} \quad \star$$

Differential eqⁿ	difference eqⁿ
cts sys	discrete sys

eg. o/p is defined by

$$c(k+1) + 2c(k) = r(k)$$

$\rightarrow \delta(k)$

Find TF

If $r(k) = \delta(k)$

Taking z transform,

$$\Rightarrow z\,C(z) - z\,C(0) + 2(C(z)) = 1$$

$$\Rightarrow (z+2)\,C(z) = 1$$

$$\Rightarrow C(z) = \frac{1}{z+2}$$

P3) ~~Multiplicⁿ Property~~

Consider the pair i.e. $\frac{z}{z-a} \leftrightarrow a^k$

or $\frac{z}{z+2} \leftrightarrow (-2)^k$.

For above $C(z)$, we want to find $c(k)$.

So, $(z^{-1}) \times \frac{z}{z+2} = C(z)$

From shifting property,

$$(z^{-1}) \times \left(\frac{z}{z+2}\right) \leftrightarrow (-2)^{k-1} \; ; k \geq 1$$

$$\text{So, } c(k) = (-2)^{k-1}$$

P3) Multiplicⁿ Property

Let $Z[f(k)] = F(z)$

Now, finding $Z[k f(k)] = \sum_{k=0}^{\infty} k f(k) z^{-k}$

Idea :- $\frac{d}{dx} x^{-n} = -n x^{-n-1}$

(creating similar form here,

$$Z[k f(k)] = \sum_{k=0}^{\infty} k f(k)(-1)(z^{-k}) \times (-1)$$

$$= \sum_{k=0}^{\infty} f(k)(-k)(z^{-k}) \times (-1)$$

$$= \sum_{k=0}^{\infty} f(k)(-k)(z^{-k}) \times \left(\frac{z^{-1}}{z^{-1}}\right) \times (-1)$$

$$= \sum_{k=0}^{\infty} f(k)(-k)(z^{-k-1}) \times \left(\frac{-1}{z^{-1}}\right)$$

$$= \frac{-1}{z^{-1}} \sum_{k=0}^{\infty} f(k)(-k z^{-k-1})$$

$$= -z \sum_{k=0}^{\infty} f(k)(-k z^{-k-1})$$

$$= -z \sum_{k=0}^{\infty} f(k) \frac{d}{dz}(z^{-k})$$

$$= (-z) \frac{d}{dz} \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$\Rightarrow \boxed{Z[k f(k)] = (-z) \frac{d}{dz} F(z)}$$

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Q) Find Z-transform of discrete ramp f^n.

$$g(k) = \begin{cases} k & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$$

We can write $g(k) = k\,u(k)$

$\Rightarrow Z[g(k)] = Z[k\,u(k)] = -z\dfrac{d}{dz}U(z)$

So, $Z[g(k)] = -z\dfrac{d}{dz}\left(\dfrac{z}{z-1}\right) = \dfrac{z}{(z-1)^2}$.

P4) Scale changing

let $Z[f(k)] = F(z)$.

Consider Z^{-1} of $F\left(\frac{z}{a}\right)$.

i.e, $Z^{-1}\left[F\left(\frac{z}{a}\right)\right] = Z^{-1}\left[\sum_{k=0}^{\infty} f(k)\left(\frac{z}{a}\right)^{-k}\right]$

$= Z^{-1}\left[\sum_{k=0}^{\infty} a^k f(k) z^{-k}\right]$

$= a^k Z^{-1}\left[\sum_{k=0}^{\infty} f(k) z^{-k}\right]$

$= a^k Z^{-1}[F(z)]$

$\Rightarrow \boxed{Z^{-1}\left[F\left(\frac{z}{a}\right)\right] = a^k f(k)}$

So, dual pair : $a^k f(k) \longleftrightarrow F\left(\frac{z}{a}\right)$

☆ Initial value & final value theorem.

	time domain	s-domain	z-domain
fn.	$f(t)$	$F(s)$	$F(z)$
Initial value	$f(t)\Big\|_{t=0}$	$\lim_{s\to\infty} sF(s)$	$\lim_{z\to\infty} F(z)$
Final Value	$f(t)\Big\|_{t\to\infty}$	$\lim_{s\to 0} \frac{F(s)}{s}$	$\lim_{k\to\infty} f(k) = \lim_{z\to 1} F(z)(z-1)$

$\Rightarrow$ final value.

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$
$$= f(0) + f(1)z^{-1} + f(2)z^{-2} + \ldots$$

Q. Consider a unity f/b integrator sys, & find value at steady state (final value)

$$C(z) = \frac{z}{z-(1-AT)} C(0) + \frac{AT}{z-(1-AT)} R(z)$$

let $r(k) = u(k)$ $(\equiv 1^k)$

$$\Rightarrow R(z) = \frac{z}{z-1}$$

Now
$$C(\infty) = \lim_{z\to 1}(z-1)C(z) = \lim_{z\to 1}\left[\frac{z(z-1)}{z-(1-AT)} C(0) + \frac{AT(z-1)}{z-(1-AT)} \underbrace{R(z)}_{\frac{z}{(z-1)}}\right]$$

$$\Rightarrow C(\infty) = 1.$$

TABLE

Property	Discrete Sequence	z - transform
① Linearity	$af(k) + bg(k)$	$aF(z) + bG(z)$
② Shifting $n \geq 0$	$f(k+n)$	$z^{n} F(z) - \sum_{i=0}^{n-1} f(i) z^{n-i}$
	$f(k-n)$	$z^{-n} F(n)$
③ Multiplic^n by k^n	$k^n f(k)$	$\left(-z \frac{d}{dz}\right)^n F(z)$
④ Scaling or multiplic^n by a^k	$a^k f(k)$	$F\left(\frac{z}{a}\right)$
⑤ Convolution	$\sum_{m=0}^{k} h(k-m) r(m)$	$H(z) R(z)$
⑥ Initial value	$f(0) = \lim_{z \to \infty} F(z)$	
⑦ Final value	$f(\infty) = \lim_{z \to 1} (1-z^{-1}) F(z)$ $= \lim_{z \to 1} (z-1) F(z)$ ↳ If $F(z)$ is analytic for $\lvert z \rvert > 1$	

☆ LAPLACE AND Z-TRANSFORMS

TABLE

	$f(t)$, $t \geq 0$	$F(s)$	$f(k)/f(kT)$, $k \geq 0$	$F(z)$
(a)	$\delta(t)$	1	$\delta(k)$	1
(b)	$u(t)$	$1/s$	$u(k)$ or 1	$\frac{z}{z-1}$
			a^k	$\frac{z}{z-a}$
			ka^k	$\frac{az}{(z-a)^2}$
			k^2a^k	$\frac{az(z+a)}{(z-a)^3}$
			$(k+1)a^k$	$\frac{z^2}{(z-a)^2}$
			$\frac{(k+1)(k+2)}{2!}a^k$	$\frac{z^3}{(z-a)^3}$
			$\frac{(k+1)(k+2)(k+3)}{3!}a^k$	$\frac{z^4}{(z-a)^4}$
			$\frac{a^k}{k!}$	$e^{az^{-1}}$

☆ Note, we get $f(k)$ by z^{-1} of ~~f~~ $F(z)$.
i.e $F(z)$ gives $f(k)$.

... continued

	$f(t)$, $t \geq 0$	$F(s)$	$f(k)$ or $f(kT)$, $k \geq 0$	$F(z)$
(c)	t	$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
(d)	t^2	$\frac{1}{s^3}$	$(kT)^2$	$\frac{T^2 z(z+1)}{(z-1)^3}$
(e)	e^{-at}	$\frac{1}{(s+a)}$	e^{-akT}	$\frac{z}{(z-e^{-aT})}$
(f)	te^{-at}	$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
(g)	$a(1-e^{-at})$	$\frac{a}{s(s+a)}$	$a(1-e^{-akT})$	$\frac{z\{1-e^{-aT}\}}{(z-1)(z-e^{-aT})}$
★ (h)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega kT$	$\frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$
★ (i)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\cos \omega kT$	$\frac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$
(j)	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-akT}\sin \omega kT$	$\frac{ze^{-aT}\sin \omega T}{\left(z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}\right)}$
(k)	$e^{-at}\cos \omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$	$e^{-akT}\cos \omega kT$	$\frac{z(z-e^{-aT}\cos \omega T)}{\left(z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}\right)}$

What will come?

★ We should know to to convert from one domain to another. And, finding initial values, final values, freq. response, time domain response of the corresponding fns.

Standard expression : $F(z) = \dfrac{b_0 z^m + b_1 z^{m-1} + \dots + b_m}{z^n + a_1 z^{n-1} + \dots + a_n}$

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* Inverse Z-transform & response of linear discrete sys :-

I) POWER SERIES METHOD :-

$$Z\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}.$$

eg Determine first few terms of the sequence $f(k)$ when

$$F(z) = \frac{z^2 + z}{z^2 - 2z + 1}$$

Applying continuous division :-

$$
\begin{array}{r l}
 & 1 + 3z^{-1} + 5z^{-2} + 7z^{-3} + \dots \\
z^2 - 2z + 1 \,\big) & z^2 + z \\
 & z^2 - 2z + 1 \\
 & - \quad + \quad - \\
\hline
 & 3z - 1 \\
 & 3z - 6 + 3z^{-1} \\
 & - \quad + \quad - \\
\hline
 & 5 - 3z^{-1} \\
 & 5 - 10z^{-1} + 5 \\
 & \vdots
\end{array}
$$

So $F(z) = 1 + 3z^{-1} + 5z^{-2} + 7z^{-3} + \dots$

$$\left.\begin{array}{l} f(0) = 1 \\ f(1) = 3 \\ f(2) = 5 \\ f(3) = 7 \\ \vdots \end{array}\right\} f(k) \text{ values}$$

(2) PARTIAL FRACTION EXPANSION METHOD

Given a f^n $F(z)\left(=\frac{Num}{den}\right)$, we can do partial fraction & can convert it into

$$F(z) = d_0 + \frac{N(z)}{M(z)} \quad ; \text{Ord } N(z) < M(z)$$

Case ① If $\exists$ distinct poles,

$$F(z) = d_0 + \frac{A_1}{z-a_1} + \frac{A_2}{z-a_2} + \dots + \frac{A_n}{z-a_n}$$

we know :- by shifting theorem,

$$\frac{z}{z-a_i} = (a_i)^k$$

$$\text{So,} \quad Z^{-1}\left(\frac{z}{z-a_i}\right) = (a_i)^{k-1}, \quad k \geq 1$$

Hence, $(a_i)^{k-1}$ will be evaluated $\forall$ partial fractions

So, we have :-

$$f(k) = d_0 \delta(k) + \underbrace{\left[A_1(a_1)^{k-1} + A_2(a_2)^{k-1} + \dots + A_n(a_n)^{k-1}\right]}_{k \geq 1}$$

Case ② If $\exists$ repeated roots (i.e., quadratic den. or higher powers)

We know $\frac{az}{(z-a)^2} \longleftrightarrow ka^k$.

We will have partial fractions of the form, say $\frac{1}{(z-a)^2}$. So,

$$\frac{1}{(z-a)^2} = \left(\frac{1}{a}\right) z^{-1}\left(\frac{az}{(z-a)^2}\right) \longleftrightarrow (k-1)a^{k-1}$$

Shifting

Case(3) If $\frac{N(z)}{M(z)}$ has a 'z' multiplied in $N(z)$ like $z\left(\frac{N(z)}{M(z)}\right)$. So, Take z outside, do partial fraction & then take inside & solve.

eg Given $F(z) = \frac{4z^2 - 2z}{z^3 - 5z^2 + 8z - 4}$

$$= \frac{4z^2 - 2z}{z^3 - 1 - 5z^2 + 5z + 3z - 3}$$

$$= \frac{4z^2 - 2z}{(z-1)(z^2+z+1) - 5z(z-1) + 3(z-1)}$$

$$= \frac{4z^2 - 2z}{(z-1)\left[z^2 + z + 1 - 5z + 3\right]}$$

$$\Rightarrow F(z) = \frac{4z^2 - 2z}{(z-1)(z-2)^2} = z\left[\frac{4z-2}{(z-1)(z-2)^2}\right]$$

$$= \frac{A}{z-1} + \frac{B}{(z-2)} + \frac{C}{(z-2)^2}$$

Solving :- $A = 2,\ B = 2,\ C = 12$

Now taking $\mathcal{Z}^{-1}$.

$$Z^{-1}\left[z\left(\frac{2}{z-1}+\frac{12}{(z-2)^2}+\frac{2}{(z-2)}\right)\right]$$

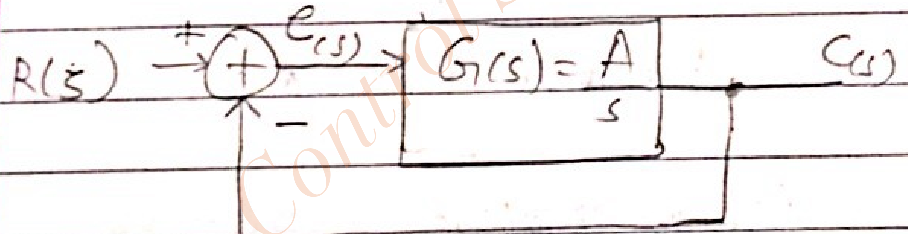
$$\Rightarrow Z^{-1}\left[\frac{2z}{z-1}+\frac{12z}{(z-2)^2}+\frac{2z}{z-2}\right]$$

$$\Rightarrow f(k) = 2[1]^{k-1}+2(2$$

Consider unity f/b integrator circuit, we have

$$C(z) = \frac{AT}{z-(1-AT)}R(z)$$

assuming zero initial cond^ns



Taking z^{-1} will give discrete time domain response.

For $u(t)$ as i/p, $R(z) = \frac{z}{z-1}$.

So, $C(z) = \frac{ATz}{[z-(1-AT)][z-1]}$

$$\Rightarrow C(z) = \frac{z}{z-1} - \frac{z}{z-(1-AT)}$$ (Partial fraction)

Taking Z^{-1}

$\Rightarrow$ $\boxed{c(k) = 1 - (1-AT)^k}$

Q. Given i/p - o/p relⁿ of a discrete time sys. given by difference eqⁿ:-

$x(k+2) - 3x(k+1) + 2x(k) = 4^k$

$x(0) = 0, \; x(1) = 1$

explains dynamics of discrete time sys.

Find solⁿ of this eqⁿ (i.e., find $x(k)$)

Taking Z - transformⁿ

$$Z[x(k+2) - 3x(k+1) + 2x(k)] = Z[4^k]$$

$$\{z^2 X(z) - z^2 x(0) - z x(1)\} - 3\{z X(z) - z x(0)\} + 2X(z) = \frac{z}{z-4}$$

$$\Rightarrow (z^2 - 3z + 2) X(z) = z^2 x(0) + z[x(1) - 3x(0)] + \frac{z}{z-4}$$

$$\Rightarrow X(z) = \underbrace{\frac{z^2 x(0) + z[x(1) - 3x(0)]}{z^2 - 3z + 2}}_{\text{Natural response}} + \underbrace{\frac{z}{(z-4)(z^2-3z+2)}}_{\text{Forced}}$$

Substituting initial cond^n ($x(0)=0$ & $x(1)=1$)

$$X(z) = \frac{z}{(z-1)(z-2)} + \frac{z}{(z-1)(z-2)(z-4)}$$

$$= \left[\frac{-z}{z-1} + \frac{z}{z-2}\right] + \left[\frac{1}{3}\left(\frac{z}{z-1}\right) - \frac{1}{2}\left(\frac{z}{z-2}\right) + \frac{1}{6}\left(\frac{z}{z-4}\right)\right]$$

Taking z^{-1}

$$\Rightarrow x(k) = \underbrace{\left(-1 + 2^k\right)}_{\text{Natural response}} + \underbrace{\left[\frac{1}{3} - \frac{1}{2}(2)^k + \left(\frac{1}{6}\right)4^k\right]}_{\text{Forced response}}$$

☆ Z-Transfer fn (PULSE TF)

↳ Impulse response for discrete time sys is called WEIGHTING SEQUENCE.

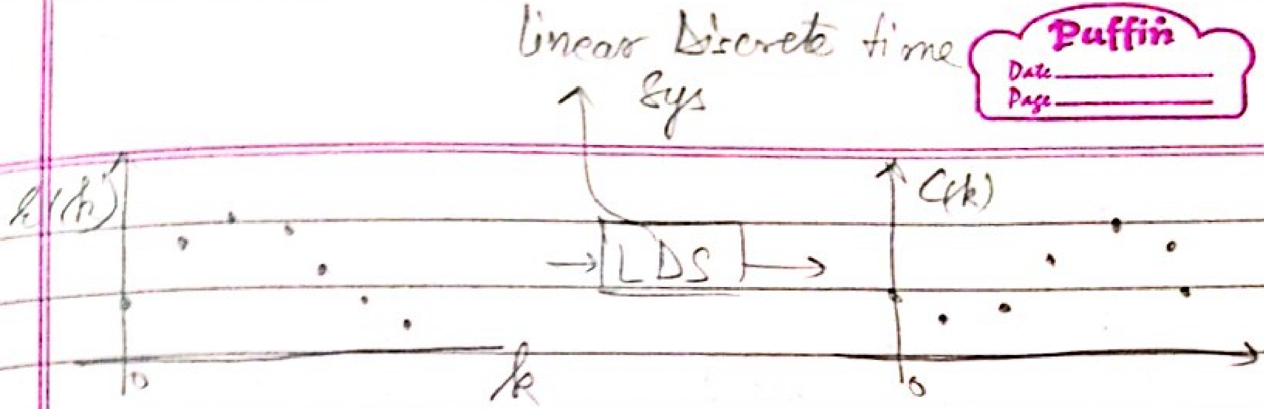
↳ $\delta(k) = \begin{cases} 1 & ; k=0 \\ 0 & ; k \neq 0 \end{cases}$

for delayed impulse response,

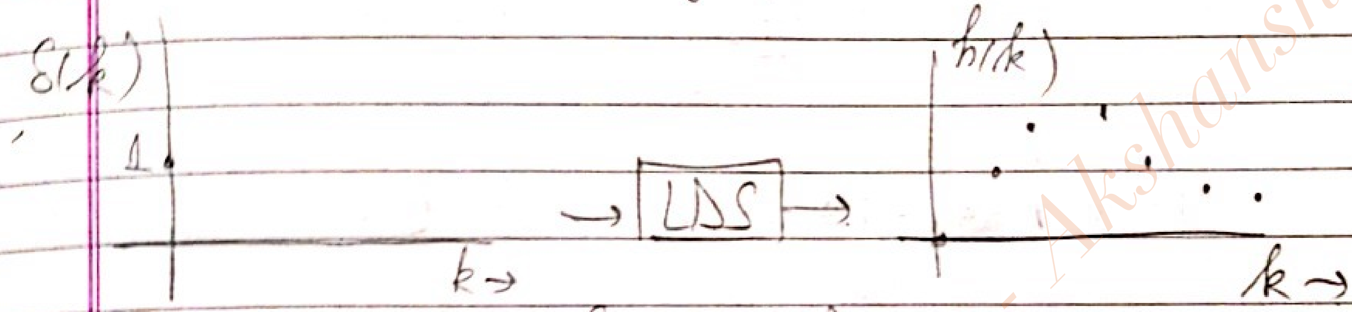
$$\delta(k-m) = \begin{cases} 1 & ; k=m \\ 0 & ; k \neq m \end{cases}$$

We are considering sampled data sys. in which sys. is cts, but i/p & o/p signal is discrete.

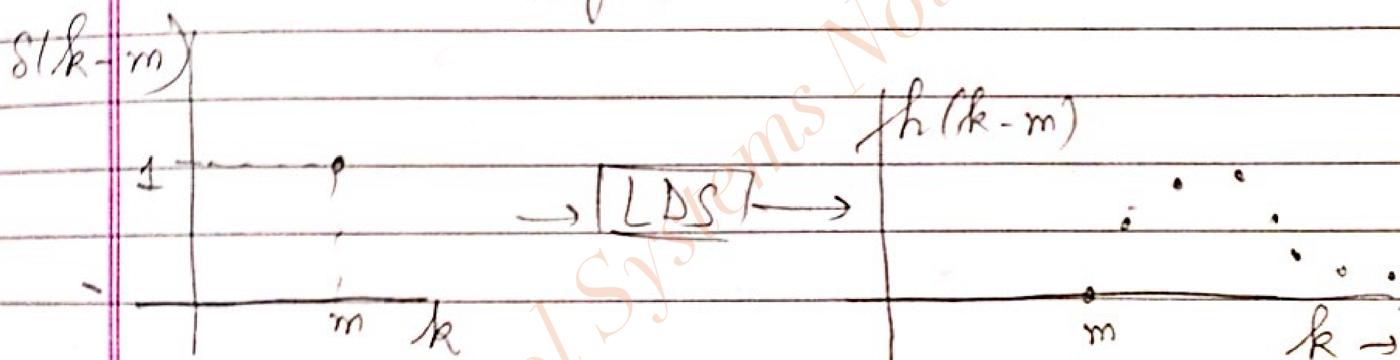
So, we have :- $r(k) \longrightarrow \boxed{LDS} \longrightarrow c(k)$



(a) Arbitrary i/p



(b) Applying δ(k) as i/p



(c) Applying shifted impulse = δ(k-m)

Let i/p sequence be expressed as sum of impulses :-

$r(k) = r(0)\,\delta(k) + r(1)\,\delta(k-1) + \cdots$

So, o/p corresponding to m^{th} sample,

$c_m(k) = r(m)\,h(k-m)$

So, total o/p, $c(k) =$

$$c(k) = \sum_{m=0}^{\infty} c_m(k) = \sum_{m=0}^{\infty} r(m)\,h(k-m)$$

(Convolution : $\sum_{\tau=0}^{\infty} f(\tau)\,g(t-\tau)$)

For discrete sys, convolution is represented by dot.

So,

$$c(k) = r(k) \cdot h(k)$$

Now, we need to consider m only from 0 to k (not 0 to ∞, as sys. is causal).

So,

$$c(k) = \sum_{m=0}^{k} r(m)\, h(k-m)$$

Proving Commutative Property of Discrete convolution.

let $j = k - m$.

$\Rightarrow m = k - j$

So, $$c(k) = \sum_{j=k}^{0} r(k-j)\, h(j)$$

$$= \sum_{j=0}^{k} h(j)\, r(k-j)$$

$\Rightarrow c(k) = h(k) \cdot r(k)$

So, $r(k) \cdot h(k) = h(k) \cdot r(k)$

So, Convolution is Commutative.

FINDING TF GIVEN DYNAMICS OF A SYS

Discrete sys: represented by Difference eq^n.

Analysis done: by Z transform. Finding TF of sys:

$$r(k) \rightarrow \underset{\text{LDS}}{\boxed{h(k)}} \rightarrow c(k) \qquad h(k) \rightarrow \underset{\text{LDS}}{\boxed{r(k)}} \rightarrow c(k)$$

let us take Z-transform of convolution sum.

So, $$Z[c(k)] = C(z) = \sum_{k=0} c(k)\, z^{-k}$$

Sys. dynamics: Rel^n b/w i/p & o/p

C(k) is defined like this when ∃ sampling & holding, i.e. when discrete sys dynamics is seen

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$$\Rightarrow C(z) = \sum_{k=0}^{\infty}\left[\sum_{m=0}^{\infty} h(k-m)\, r(m)\right] z^{-k}$$

Interchanging order of summation,

$$C(z) = \sum_{m=0}^{\infty} r(m) \sum_{k=0}^{\infty} h(k-m)\, z^{-k}$$

let $j = k-m$, then

$$C(z) = \sum_{m=0}^{\infty} r(m) \sum_{j=-m}^{\infty} h(j)\, z^{-j-m}$$

$$\Rightarrow C(z) = \underbrace{\sum_{m=0}^{\infty} r(m) z^{-m}}_{R(z)} \underbrace{\sum_{j=0}^{\infty} h(j)\, z^{-j}}_{H(z)}$$

$$\Rightarrow C(z) = R(z)\, H(z)$$

$$\Rightarrow TF = \frac{C(z)}{R(z)} = H(z) \qquad R(z) \rightarrow \boxed{H(z)} \rightarrow C(z)$$

When i/p = impulse f^n ($\delta(k)$),

$\Rightarrow r(k) = \delta(k)$

$\Rightarrow R(z) = Z\{\delta(k)\} = 1$

So, now, $TF = \frac{C(z)}{1} = H(z)$

So, when i/p = impulse f^n, o/p or response, $H(z) = C(z)$

So, unit impulse response of sys (discrete) $= C(z)$

Taking Z^{-1} :- In time domain, discrete impulse response $= c(k) \quad (Z^{-1}(C(z)))$

Consider another sys :-

$R(z) \rightarrow \boxed{H(z)} \xrightarrow{D(z)} \boxed{G(z)} \rightarrow C(z)$

$\equiv$

$R(z) \rightarrow \boxed{H(z)G(z)} \rightarrow C(z)$

Let the difference eq^n is given as :

$$c(k) + a_1 c(k-1) + \ldots a_n c(k-n) = b_0 r(k) + b_1 r(k-1) + \ldots + b_n r(k-n)$$

$Z\{z^{-1}C(z)\}$

Taking Z transform :-

$$C(z) + a_1 z^{-1} C(z) + \ldots + a_n z^{-n} C(z) = b_0 R(z) + b_1 z^{-1} R(z) + \ldots + b_n z^{-n} R(z)$$

$$\Rightarrow C(z)\left[1 + a_1 z^{-1} + \ldots + a_n z^{-n}\right] = R(z)\left[b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}\right]$$

$$\Rightarrow \frac{C(z)}{R(z)} = TF = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-n}}$$

Imp Understanding Concept : In ct domain, I am given a sys, say a circuit. I apply KCL, KVL & get eq^{ns}. These eq^{ns} are integro differential eq^{ns}. I take LT of it, group C(s) & R(s) terms. Then, $TF = \frac{C(s)}{R(s)}$. Illy, in Z-domain, the eq^n I have is called difference eq^n. I take Z-transform, group C(z) & R(z) to get $TF = H(z) = C(z)/R(z)$

So,

$$H(z) = \frac{b_0 z^n + b_1 z^{n-1} + \dots + b_n}{z^n + a_1 z^{n-1} + \dots + a_n}$$

* Finding impulse response, $R(z) = 1$

So, $H(z) = Z\left\{ \frac{C(k)}{r(k)} \right\}$.

$r(k) \to \delta(k)$

Q. Determine impulse response (weighting sequence) of LDS, described by:

$$C(k) - \alpha\, c(k-1) = r(k)$$

Taking z-transform,

$$C(z) - \alpha z^{-1} C(z) = R(z)$$

$$\Rightarrow C(z)\left[1 - \alpha z^{-1}\right] = R(z)$$

$$\Rightarrow \frac{C(z)}{R(z)} = \frac{1}{1 - \alpha z^{-1}} \longrightarrow \left(= \frac{a}{1-r} \right)$$

Sum of infinite GP

$$H(z) = \frac{C(z)}{R(z)} = \frac{z}{z-\alpha}.$$

Taking impulse response $\Rightarrow R(z) = 1$.

$$\Rightarrow H(z) = C(z) = \frac{z}{z-\alpha}.$$

or $h(k) = \begin{cases} \alpha^k & ; k \geq 0 \\ 0 & ; k \leq 0 \end{cases}$ for Causal i/p.

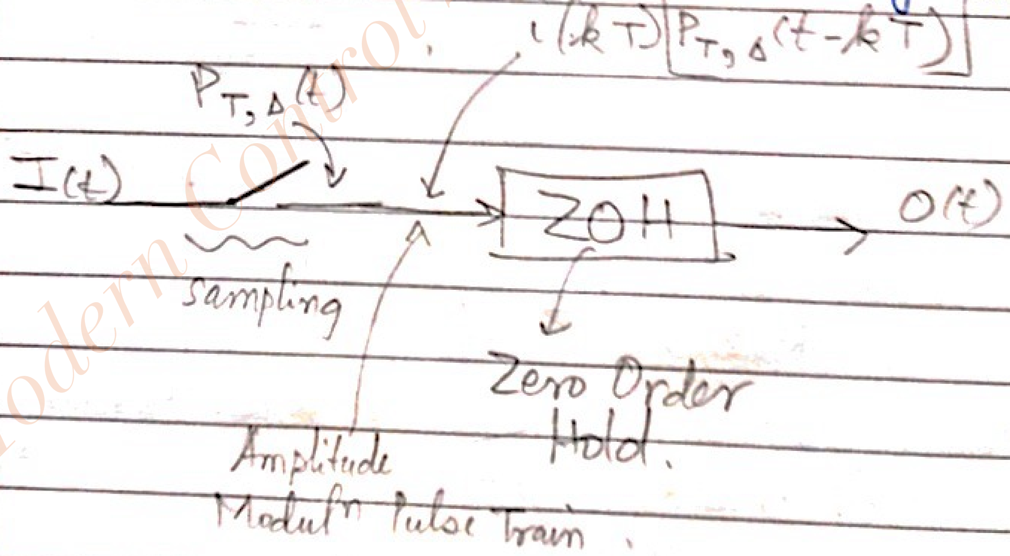
Determining initial cond^ns for higher ord. sys :-

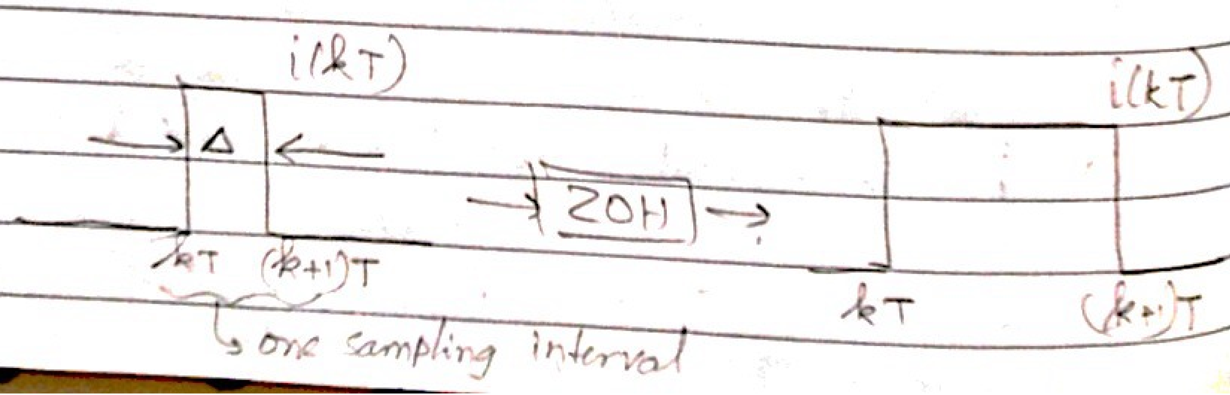
If $c(k)=0$ & $r(k)=0$ for $k<0$ (causal i/p)

Suppose, while finding TF, while taking Z-transform, we get initial cond^n terms like $r(1)$, $r(2)$, $c(0)$, $c(1)$, $c(2)$ --- etc, how to find them?

We have the difference eq^n given.
Suppose, highest order is 2 { i.e. I have $c(k+2)$ term. If I want to find $c(1)$, put $k=-1$
So, I get $c(1)+a_1 c(0) = b_0 r(1) + b_1 r(0)$
I'ly, find all initial cond^ns

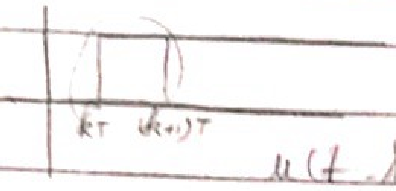
MODELLING OF Discrete Time System





o/p of ZOH can be written as:-

$$O(t) = \sum_{k=0}^{\infty} i(kT)\underbrace{\left[u(t-kT) - u(t-\overline{k+1}\,T)\right]}$$



$u(t-kT) - u(t-\overline{k+1}\,T)$

Taking LT,

$$\Rightarrow O(s) = \sum_{k=0}^{\infty} i(kT)\left[\frac{e^{-skT} - e^{-s(k+1)T}}{s}\right]$$

$$\Rightarrow O(s) = \left(\frac{1-e^{-sT}}{s}\right)\underbrace{\sum_{k=0}^{\infty} i(kT)e^{-skT}}$$

Taking $\mathcal{L}^{-1}$ of

$$\Rightarrow \mathcal{L}^{-1}\left(\sum_{k=0}^{\infty} i(kT)e^{-skT}\right) = \sum_{k=0}^{\infty} i(kT)\delta(t-kT) = \underbrace{i(t)\,\delta_T(t)}$$

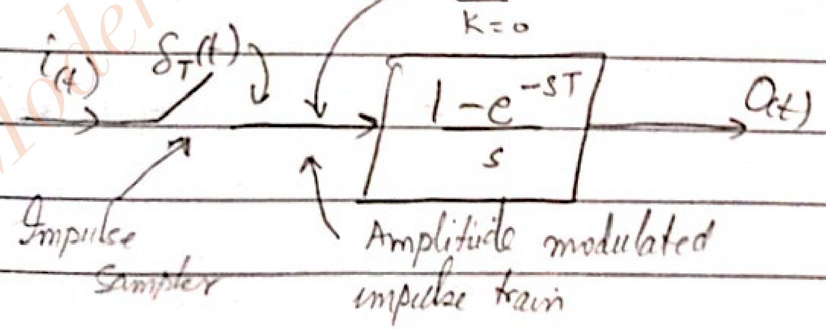
$\Downarrow$

Cts signal multiplied with impulse fn

$\Downarrow$

Sampling & holding

$$\sum_{k=0}^{\infty} i(kT)\,\delta(t-kT) = i^*(t)$$



Sampling & holding oper^n : replaced by s-domain TF.

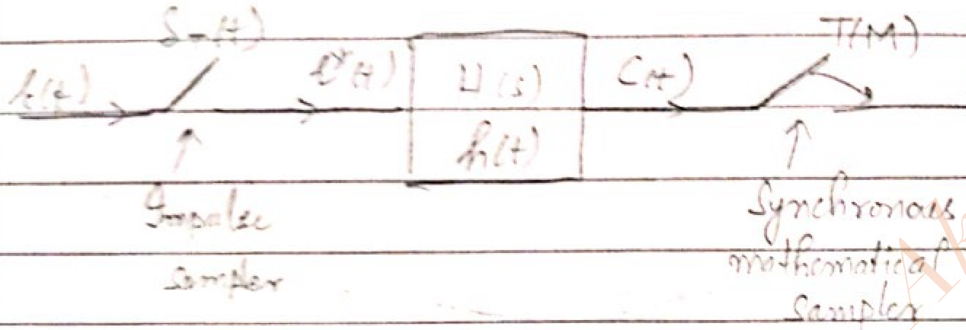
We have s-domain & we need to go to z-domain

Idea :- Take $\mathcal{L}^{-1}$ to get time domain Replace $t \rightarrow kT$ to get discrete time domain. Take Z-transform to get Z dom.

* Different cases:

1. Continuous reference i/p.

Given: $C(s) = H(s) \cdot R^*(s)$; $R(s) = \mathcal{L}[\underbrace{r^*(t)}_{\text{Impulse sampled input signal}}]$



$\mathcal{L}^{-1}[H(s)] = h(t)$

Now

$$r^*(t) = \sum_{k=0}^{\infty} r(kT)\,\delta(t-kT)$$

Using superposⁿ,

$$c(t) = \sum_{k=0}^{\infty} r(kT)\,h(t-kT)$$

$$c(nT) = \sum_{k=0}^{\infty} r(kT)\,h(nT-kT)$$

$$H(z) = Z[\underbrace{H(s)}_{\text{found from time domain}}]$$

eg $H(s) = \dfrac{a}{s(s+a)} = \dfrac{1}{s} - \dfrac{1}{s+a}$

Taking L^{-1}

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+a}\right] = \left[1 - e^{-at}\right]u(t)$$

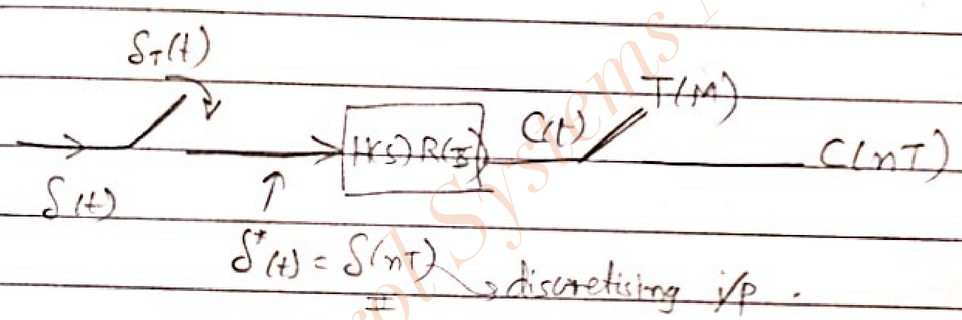
$$t \to kT \Rightarrow h(kT) = \begin{cases} 1 - e^{-akT} & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$$

Taking z-transform:

$$H(z) = \frac{z}{z-1} - \frac{z}{z-e^{-aT}} = \frac{z\{1-e^{-aT}\}}{(z-1)(z-e^{-aT})}$$

☆ If there exists a sample & hold at feedback at output, then, we are considering digital signals

2.



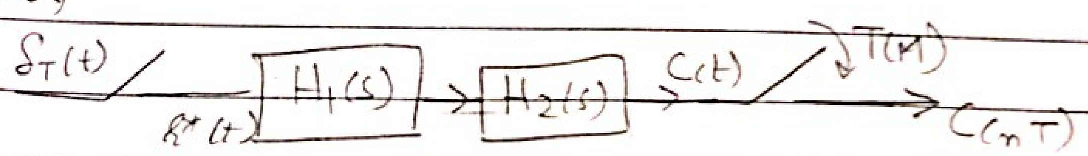
$$C(s) = H(s)R(s) \longleftrightarrow C(z) = HR(z)$$

So,

$$\boxed{Z[H(s)R(s)] = HR(z)}$$ i.e. $Z[h(k) * r(k)]$

☆ $(\neq H(z) \cdot R(z))$ $Z[h(k)] \cdot Z[r(k)]$

So,



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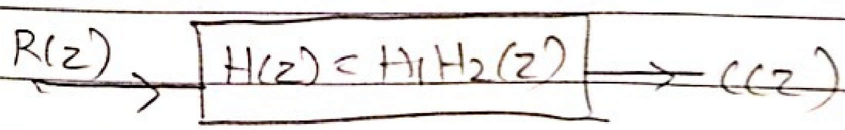
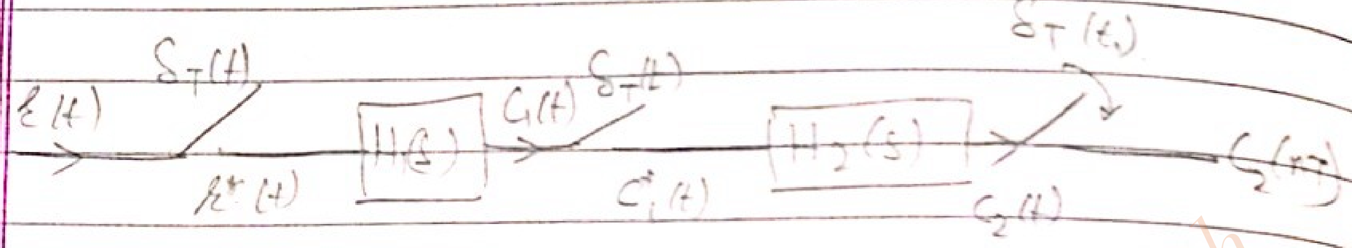
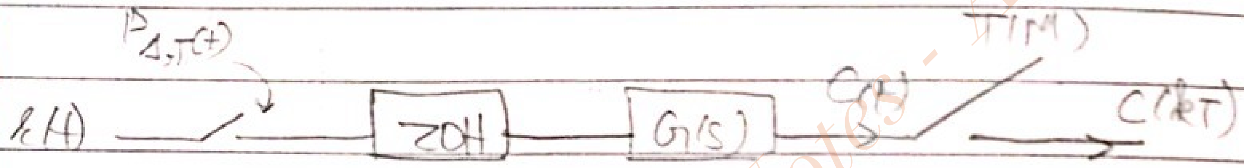


3. Now, discretising o/p$_s$ of TF



4. Now, alongwith sampling, putting hold also:



conclusion (on seeing block diagrams):-
ZOH introduces single delay :-

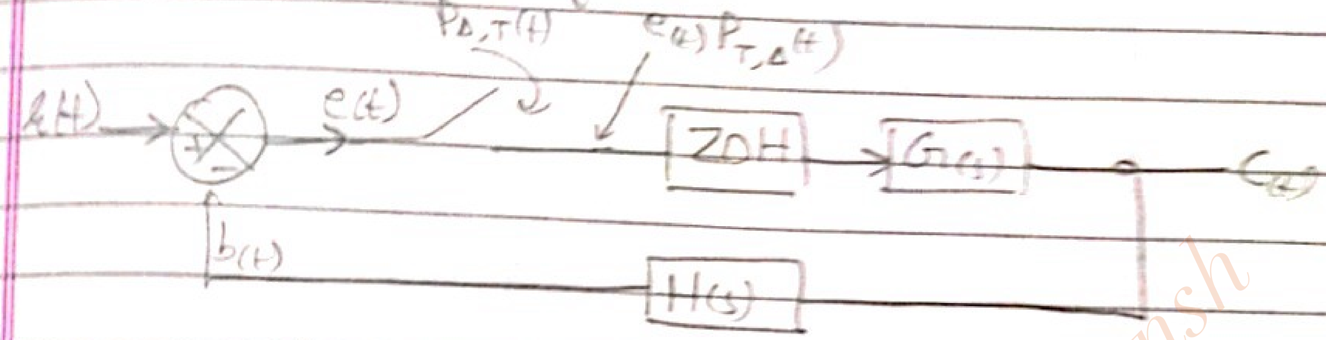
$$Z\left[G_0(s)G(s)\right] = (1-z^{-1})\,Z\left[\frac{G(s)}{s}\right]$$

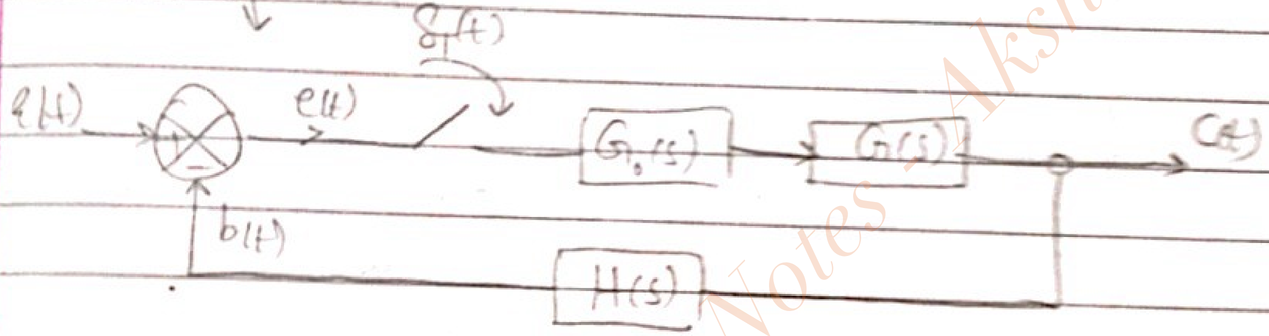
eg:- $G(s) = \dfrac{a}{s+a}$

So, $Z\left[\dfrac{G(s)}{s}\right] = Z\left[\dfrac{a}{s(s+a)}\right] = \dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$

(from table)

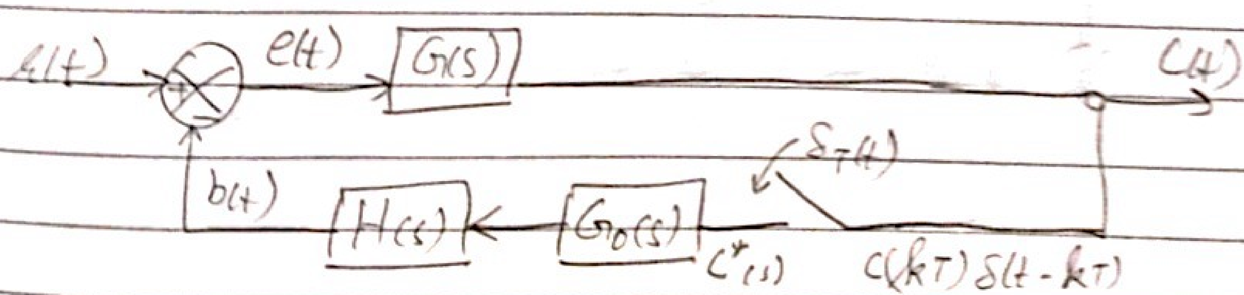
5. ✫ Now seeing sampling in CLTF





$$C(z) = Z[G_0(s)G(s)]E(z)$$
$$B(z) = Z[G_0(s)\,G(s)H(s)]E(z)$$
$$e(t) = r(t) - b(t)$$
$$e(kT) = r(kT) - b(kT)$$
$$E(z) = R(z) - B(z) \Rightarrow E(z)[1 + Z[G_0(s)G(s)H(s)]] = R(z)$$
$$\Rightarrow E(z) = \frac{R(z)}{1 + Z[G_0GH]}$$

6. ✫ Sampling & holding in CLTF : feedback



$$C(z) = \frac{RG(z)}{1 + G_0HG(z)}$$

☆ Z-domain & s-domain relationship.

Z-Transform of $r(t)$ $\quad R(z) = \sum_{k=0}^{\infty} r(kT)\, z^{-k}$

Discretized signal $\quad r^*(t) = \sum_{k=0}^{\infty} r(kT)\, \delta(t-kT)$ (Convolution)

LT of $r^*(t)$ $\quad R^*(s) = \sum_{k=0}^{\infty} r(kT)\, e^{-ksT}$

$\hookrightarrow$ If $e^{sT} = z$ or $s = \frac{1}{T} \ln z$

$\Rightarrow R^*(s) = \sum_{k=0}^{\infty} r(kT)\, z^{-k} = R(z)$

$\Rightarrow$ Laplace Transform of discrete time signal $=$ Z transform of discrete time signal

$\hookrightarrow$ by $\boxed{s = \frac{1}{T} \ln z}$ ☆ $\qquad \hookrightarrow$ by $\boxed{z = e^{sT}}$ ☆

Now,

$z = e^{sT}$

$= e^{(\sigma + j\omega)T}$

taking only complex part

$= e^{j\omega T}$

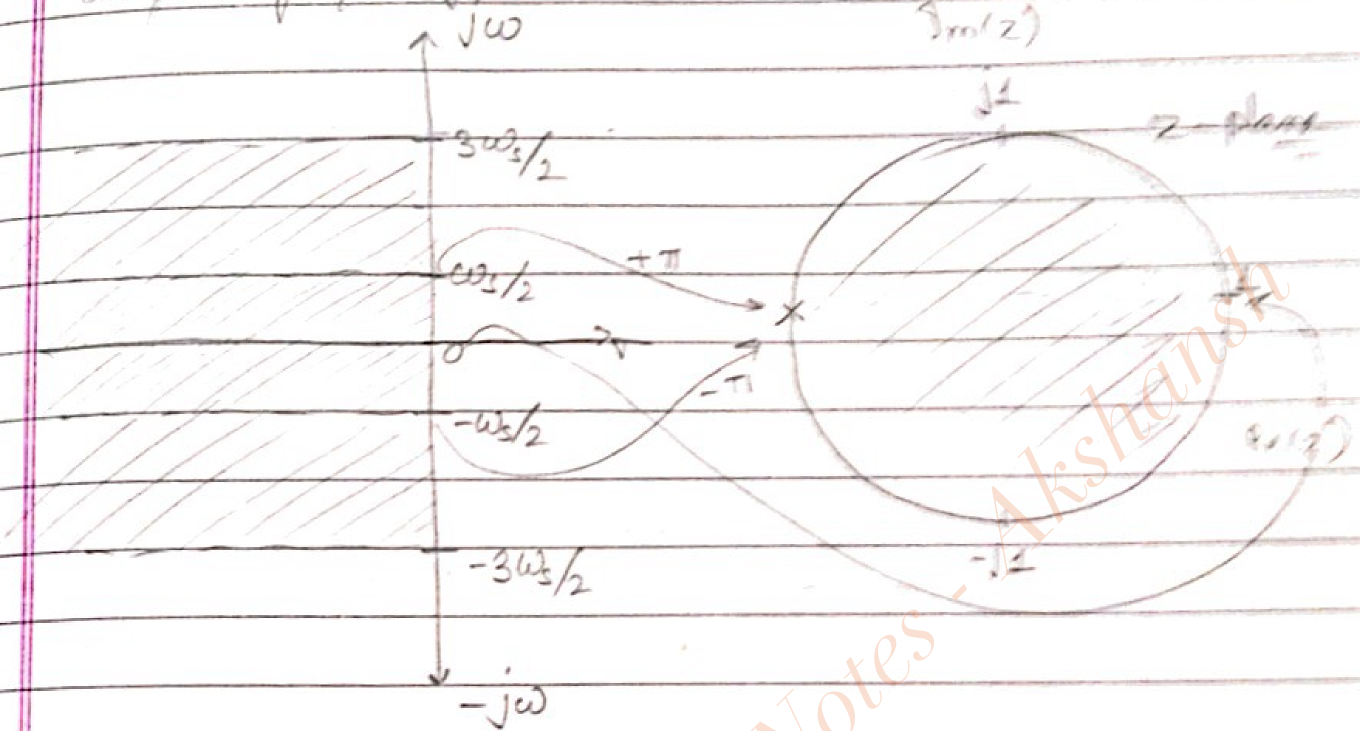
$= e^{j\omega(2\pi/\omega_s)}$ $\quad \left(\because \omega_s = \frac{2\pi}{T}\right)$

$= e^{j2\pi(\omega/\omega_s)}$

$\Rightarrow z = 1 \angle 2\pi(\omega/\omega_s)$ (in polar form)

Idea :- we know stability criterion in s-domain (Routh Hurwitz criterion, Bode plot, root locus technique). We want to see that in Z-domain. So, find relⁿ b/w s & z domain.

s-plane : giving complete freq range



Substitute $\omega \to \frac{\omega_s}{2}$

So, $Z = e^{j2\pi\left(\frac{\omega_s}{2}\right)\times\frac{1}{\omega_s}}$

$= e^{j\pi}$

$Z = -1, \theta = +\pi$

$\omega \to -\frac{\omega_s}{2}$

$Z = 1 ; \theta = -\pi$

Seeing left half of s-plane :

$S = -\sigma + j\omega$ (general represent^n)

Now, $Z = e^{ST}$

$= e^{-\sigma T} \cdot e^{j\omega T}$

$= \frac{1}{e^{\sigma T}} \; e^{j\omega T}$

$= \left(\frac{e^{j\omega T}}{e^{\sigma T}}\right) \to < 1 \Rightarrow$ always inside unit circle $\Rightarrow$ stable

So that why we say, inside unit circle = stable

Hence, poles inside unit circle : Stable
on " : Marginally stable
outside " : Unstable

Now,
doing stability analysis to see if sys is stable/no :-

For discrete sys :- $C(z) = R(z)\,T(z)$

for impulse i/p, $R(z) = 1$.
$\forall$ real poles

$$C(z) = T(z) = \frac{A_1}{z-a_1} + \frac{A_2}{z-a_2} + \dots + \frac{A_n}{z-a_n}$$

Take Z^{-1}

$$\Rightarrow c(kT) = A_1(a_1)^{k-1} + A_2(a_2)^{k-1} + \dots + A_n(a_n)^{k-1}$$
$\hookrightarrow k \geq 1$

If impulse response $\to 0$ at $t \to \infty \Rightarrow$ sys. is stable

So, $c(kT) \to 0$ as $k \to \infty$.
Condⁿ :- $|a_i| < 1$, $i = 1, 2, \dots, n$
$\hookrightarrow$ |roots|
$\hookrightarrow < 1 \Rightarrow$ Inside unit circle.

Now, if $z = a_i \pm j b_i$: complex conj. pole pair
Then, response :-
$$(\alpha_i)^k \cos[k\theta_1 + \phi_i]$$
$\hookrightarrow \alpha_i = \sqrt{a_i^2 + b_i^2}$: decays for $\alpha_i < 1$
$\phi_i = \tan^{-1}\left(\frac{b}{a}\right)$

Illy, $\frac{1}{(z-a_i)^2} \longleftrightarrow (k-1)\, a_i^{k-2}, \; k \geq 2$

↳ also decays for $\alpha_i < 1$

Now,

Seeing stability by Char. eq^n of sampled data sys:-

$$1 + Z\left[G_0(s) G(s) H(s)\right] = 0$$

If discretised sys. is used in cascade with above sampled data sys, we have

$$1 + D(z)\, Z\left[G_0(s) G(s) H(s)\right] = 0$$

- **Methods for Stability Analysis :-**

Char. polynomial $F_1(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 = 0$

↳ $a_n > 0$

Positive powers of z

JURY'S TEST (like Routh's Criterion in s-domain)

necessary cond^{ns} :-

✓ $F_1(1) > 0$

✓ $(-1)^n F_1(-1) > 0$

Sufficient cond^n :- (from Jury's Table)

PTO

Total rows = $2n-3$

Row	z^0	z^1	z^2	z^3	...	z^{n-k}	...	z^{n-2}	z^{n-1}	z^n
1 Roots in order	a_0	a_1	a_2	a_3	...	a_{n-k}	...	a_{n-2}	a_{n-1}	a_n
2 Roots reverse order	a_n	a_{n-1}	a_{n-2}	a_{n-3}	...	a_k	...	a_2	a_1	a_0
3 Determinant of 1st two & last two	b_0	b_1	—	—	—	—				
4 Reverse order	b_n	b_{n-1}	—	—	—	—				
⋮										
$2n-4$	s_3	s_2	s_1	s_0						
$2n-3$	r_0	r_1	r_2							

b_0 → Determinant of 1st two & last two

b_1 → determinant of 1st two & second last two

Sufficient cond^n:

$$\left.\begin{aligned} |a_0| &< |a_n| \\ |b_0| &> |b_{n-1}| \\ |c_0| &> |c_{n-2}| \\ &\vdots \\ |r_0| &> |r_2| \end{aligned}\right\} (n-1) \text{ constraints}$$

eg. $F_1(z) = 2z^4 + 7z^3 + 10z^2 + 4z + 1$.

Check whether sys. is stable or not.

Necessary condⁿ

$$F_1(1) = 2(1)+7(1)+10(1)+4(1)+1 > 0$$

Satisfied

$$(-1)^4 F_1(-1) = 2-7+10-4+1 = 2 > 0$$

Satisfied

2) Sufficient condⁿ :-

Row	z^0	z^1	z^2	z^3	z^4
1	1	4	10	7	2
2	2	7	10	4	1
3	-3	-10	-10	-1	
	$\begin{vmatrix}1 & 2\\2 & 1\end{vmatrix}$	$\begin{vmatrix}1 & 7\\2 & 4\end{vmatrix}$	$\begin{vmatrix}1 & 10\\2 & 10\end{vmatrix}$	$\begin{vmatrix}1 & 4\\2 & 7\end{vmatrix}$	
4	-1	-10	-10	-3	
5 (2n-3)	8	20	20		
	$\begin{vmatrix}-3 & -1\\-1 & -3\end{vmatrix}$	$\begin{vmatrix}-3 & -10\\-1 & -10\end{vmatrix}$	$\begin{vmatrix}-3 & -10\\-1 & -10\end{vmatrix}$		

So, for stability :-

$|1| < |2|$ ✓

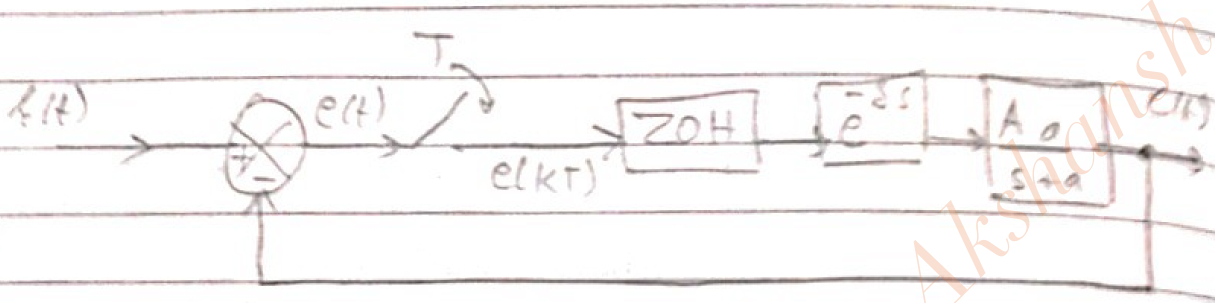
$|-3| > |-1|$ ✓

$|8| > |20|$ X → not satisfied

So, unstable.

eg: A sampled data control sys. of order 1 with transport" lag is shown. Determine cond" for sys stability. $\delta < T$.

Sol": Char. eq'n $F_1(z) = 1 + Z[G_0(s)\, G(s) H(s)]$



$$G(s) = \underbrace{\left(\frac{1-e^{-sT}}{s}\right)}_{G_0(s)} e^{-\delta s}\left(\frac{A_0}{s+a}\right) = \left[\frac{e^{-\delta s} - e^{-(T+\delta)s}}{s}\right]\frac{A_0}{s+a}$$

$$\Rightarrow G(s) = A\left[e^{-\delta s} - e^{-(T+\delta)s}\right]\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

Finding $G(z)$: $\longrightarrow \mathcal{L}^{-1}$ of $G(s)$

$t \to kT$

Z-transform

$$\mathcal{L}^{-1}[G(s)] = g(t) = A[1-e^{-a(t-\delta)}]u(t-\delta) - A[1-e^{-a(t-T-\delta)}]u(t-T-\delta)$$

$t \to kT$

$$\Rightarrow g(kT) = A[1-e^{-a(kT-\delta)}]u(kT-\delta) - A[1-e^{-a(kT-T-\delta)}]u(kT-T-\delta)$$

Z-transform

$$G(z) \neq (1-z^{-1})\, Z\left\{\frac{A a e^{-\delta s}}{s(s+a)}\right\}$$

$$= \begin{cases} 0 & k=0 \\ A(1-e^{-a(T-\delta)}) & k=1 \\ A(1-e^{-a(kT-\delta)}] - A[1-e^{-a(kT-T-\delta)}] & k \geq 2 \\ A e^{a\delta} e^{-akT}[e^{aT}-1] & k \geq 2 \end{cases}$$

So

$$G(z) = A[1-e^{-a(T-\delta)}]z^{-1} + Ae^{a\delta}(e^{aT}-1)\sum_{k=2}^{\infty} e^{-akT} z^{-k}$$

$$= A[1-e^{-a(T-\delta)}]z^{-1} + Ae^{a\delta}(e^{aT}-1)z^{-2}\sum_{k=2}^{\infty} e^{-akT} z^{-(k-2)}$$

Put $k-2 = r$, we have

$$G(z) = A[1-e^{-a(T-\delta)}]z^{-1} + Ae^{a\delta}(e^{aT}-1)z^{-2}\sum_{r=0}^{\infty} e^{-2aT} e^{-arT} z^{-r}$$

$$= A[1-e^{-a(T-\delta)}]\frac{1}{z} + Ae^{a\delta}(e^{aT}-1)e^{-2aT}\left(\frac{1}{z(z-e^{-aT})}\right)$$

$$G(z) = \frac{A[1-e^{-a(T-\delta)}][z-e^{-aT}] + e^{a\delta}e^{-aT}(1-e^{-aT})}{z(z-e^{-aT})}$$

Char. eq^n:-

$$F_1(z) = 1 + G(z)$$

$$= z(z-e^{-aT}) + A[1-e^{-a(T-\delta)}](z-e^{-aT}) + Ae^{-a(T-\delta)}(1-e^{-aT})$$

$$\Rightarrow F_1(z) = \underbrace{z^2}_{a_2} + \underbrace{\{A[1-e^{-a(T-\delta)}] - e^{-aT}\}z}_{a_1} + \underbrace{Ae^{-aT}(e^{a\delta}-1)}_{a_0}$$

Necessary cond^n:-

(1) $F_1(1) > 0$ (Put $z=1$ in $F_1(z)$)

$$\Rightarrow 1 + A[1-e^{-a(T-\delta)}] - e^{-aT} + Ae^{-a(T-\delta)} - Ae^{-aT} > 0$$

$$\Rightarrow (1-e^{-aT})(1+A) > 0$$

(2) $(-1)^2 F_1(-1) > 0.$

$$\Rightarrow 1 - A + Ae^{-a(T-\delta)} + e^{-aT} + Ae^{-a(T-\delta)} - Ae^{-aT} > 0$$

$$\Rightarrow (1+e^{-aT})(1-A) + 2Ae^{-a(T-\delta)} > 0$$

$$\Rightarrow A < \frac{1+e^{-aT}}{e^{-aT}(2e^{a\delta}-1)-1}$$

Sufficient cond^n :-

$$A\left[e^{-a(T-\delta)} - e^{-aT}\right] < 1$$

$$\hookrightarrow A e^{-aT}(e^{a\delta}-1) < 1$$

Combining all constraints on A :

$$A < \left(\frac{e^{aT}}{e^{a\delta}-1}\right)$$

☆ Finding $G(z) \rightarrow$ discrete TF was difficult. Now, making it easy by using :

Routh Nyquist Criterion

Defining a new plane $\rightarrow$ r-plane, s.t $r = \frac{z-1}{z+1} \Rightarrow z = \frac{1+r}{1-r}$.

(inside unit circle $\xrightarrow{\text{by BZT}}$ left half of r-plane)

On unit circle in z plane :-

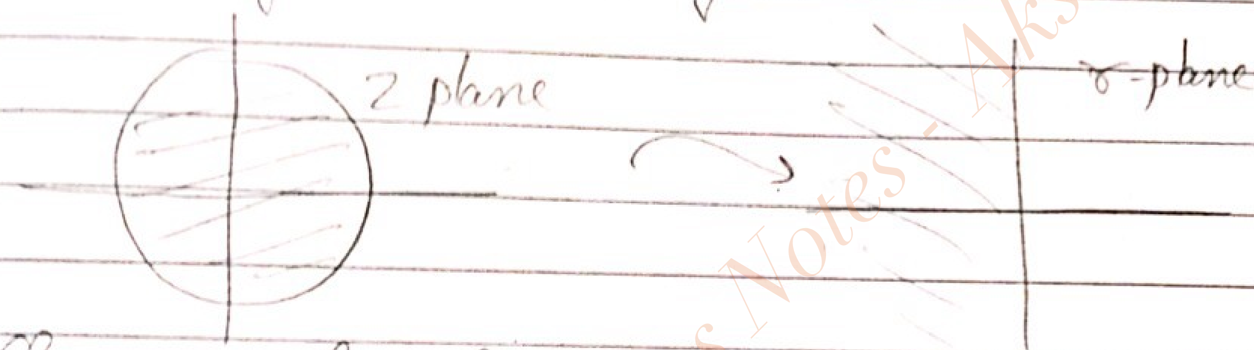
$z = e^{j\theta}$ (θ varyies counterclockwise from $-\pi$ to 0 to π)

$$\Rightarrow r = \frac{e^{j\theta}-1}{e^{j\theta}+1} = \frac{e^{j\theta/2} - e^{-j\theta/2}}{e^{j\theta/2} + e^{-j\theta/2}}$$

$\Rightarrow r = \tan j\frac{\theta}{2} = j\tan\frac{\theta}{2} = j\omega_r$

; $\omega_r = \tan\frac{\theta}{2}$ varies from $-\infty$ through 0 to $+\infty$

So, basically, we are doing



Char. eq^n (transformed in r-plane using BZT), we get / $z \to \frac{1+r}{1-r}$ in Char eq^n in linear discrete time)

$$a_n\left(\frac{1+r}{1-r}\right)^n + a_{n-1}\left(\frac{1+r}{1-r}\right)^{n-1} + \dots + a_1\left(\frac{1+r}{1-r}\right) + a_0 = 0$$

Transforming (removing denominator & expanding)

$$b_n r^n + b_{n-1} r^{n-1} + \dots + b_1 r + b_0 = 0$$

$\hookrightarrow$ If roots of this eq^n lie on left of r-plane

$\Downarrow$

Sys. is stable.

eg Consider sampled sys :-

$$G(z) = Z\left[\frac{5}{s(s-1)(s+2)}\right] = 5\,Z\left[\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right]$$

$$= 5\left[\frac{z}{2(z-1)} - \frac{z}{z-e^{-1}} + \frac{z}{2(z-e^{-2})}\right]$$

$$G(z) = \frac{5z(0.4z + 0.594)}{2(z-1)(z-0.368)(z-0.135)}$$

Char. eq^n is :-

$$1 + G(z) = 0$$

$$\Rightarrow 2(z-1)(z-0.368)(z-0.135) + 5z(0.4z+0.594) = 0$$

$$\Rightarrow z^3 - 0.5z^2 + 2.49z - 0.496 = 0$$

$$z \rightarrow \frac{1+k}{1-k}$$

$$\Rightarrow \left(\frac{1+k}{1-k}\right)^3 - 0.5\left(\frac{1+k}{1-k}\right)^2 + 2.49\left(\frac{1+k}{1-k}\right) - 0.496 = 0$$

$$\Rightarrow \underbrace{3.5k^3 - 2.5k^2 + 0.5k + 2.5 = 0}$$

↳ coeff. are having change in sign.

$\Rightarrow$ by Routh Criterion (done in control sys)

sys is unstable

Aliter :- Seeing from $G(s)$ (like we did in Control Sys)

For ct time sys :- $1 + G(s) = 0$.

$\Rightarrow 1 + \frac{5}{s(s+1)(s+2)} = 0.$

$\Rightarrow s^3 + 3s^2 + 2s + 5 = 0$

Routh array

s^3	1	2
s^2	3	5
s^1	$1/3$	0
s^0	5	

$-\frac{\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}}{3}$

no sign change

$\Rightarrow$ sys is stable

Inference : Stable cts sys can get changed to unstable when its discretised.

✯ Root Locus Technique

$$1 + F(z) = 0$$

$$\rightarrow 1 + \frac{K \prod(z+z_i)}{\prod(z+p_j)} = 0$$

eg Analyse stability for sampling period = 0.4 sec & 3 sec.

Here,

$$G_0(s) = \frac{1 - e^{-sT}}{s}, \quad G(s) = \frac{K}{s(s+2)}$$

So, Char. eq^n :- $1 + G_0G(z) = 0.$

$$G_0G(z) = Z\left[\frac{1 - e^{-sT}}{s} \cdot \frac{K}{s(s+2)}\right] = 0$$

solving - -

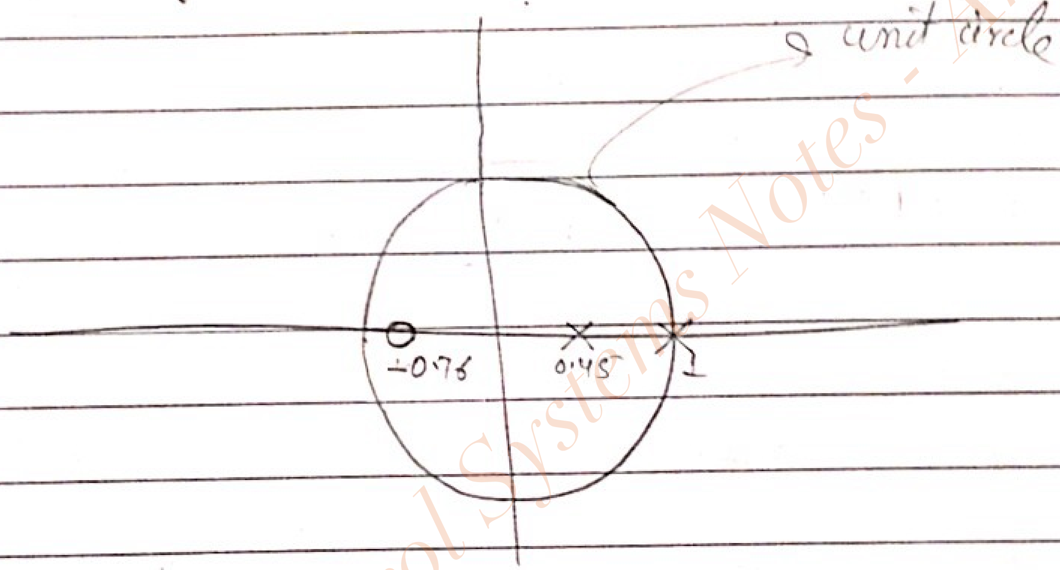
Taking T = 0.4 sec ..

$$G_0G(z) = \frac{K(z+0.76)}{16(z-1)(z-0.45)} = \frac{K'(z+0.76)}{(z-1)(z-0.45)}$$

So, zero : -0.76

Poles : $1, 0.45$

Plotting root locus:



Finding Breakaway pts:-

$$1 + \frac{K'(z+0.76)}{(z-1)(z-0.45)} = 0$$

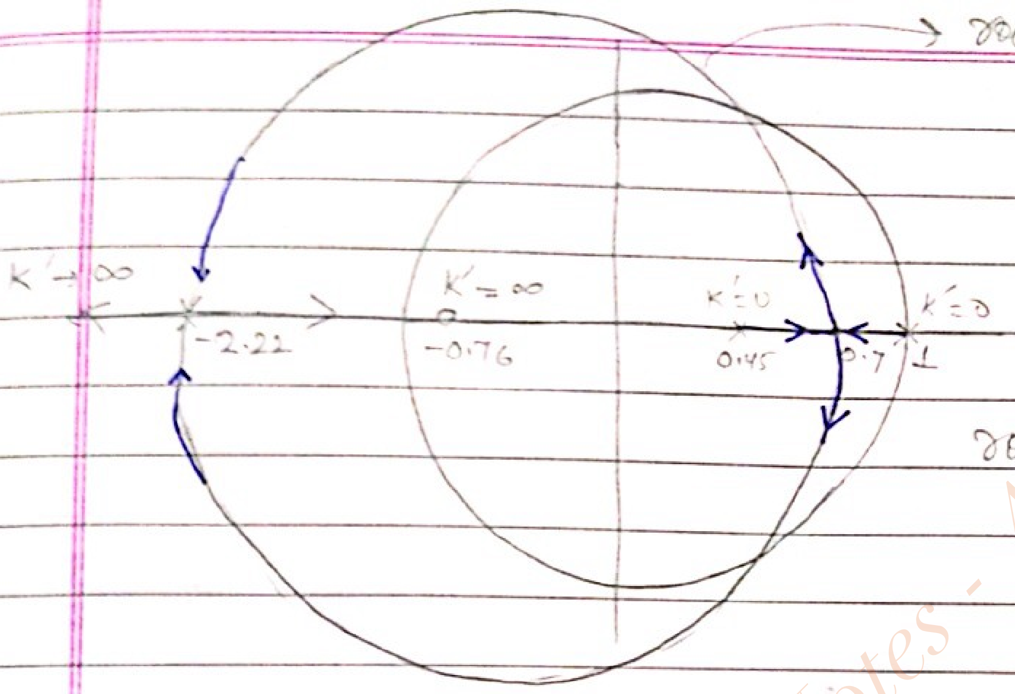
$$K' = -\frac{(z-1)(z-0.45)}{(z+0.76)}$$

$$\frac{dK'}{dz} = 0 \Rightarrow \frac{(z^2-1.45z+0.45) - (z+0.76)(-2z+1.45)}{(z+0.76)^2}$$

$$\Rightarrow z^2+1.52z-1.55 = 0$$

$$\Rightarrow z = -2.22, 0.7$$

breakoff at 0.7

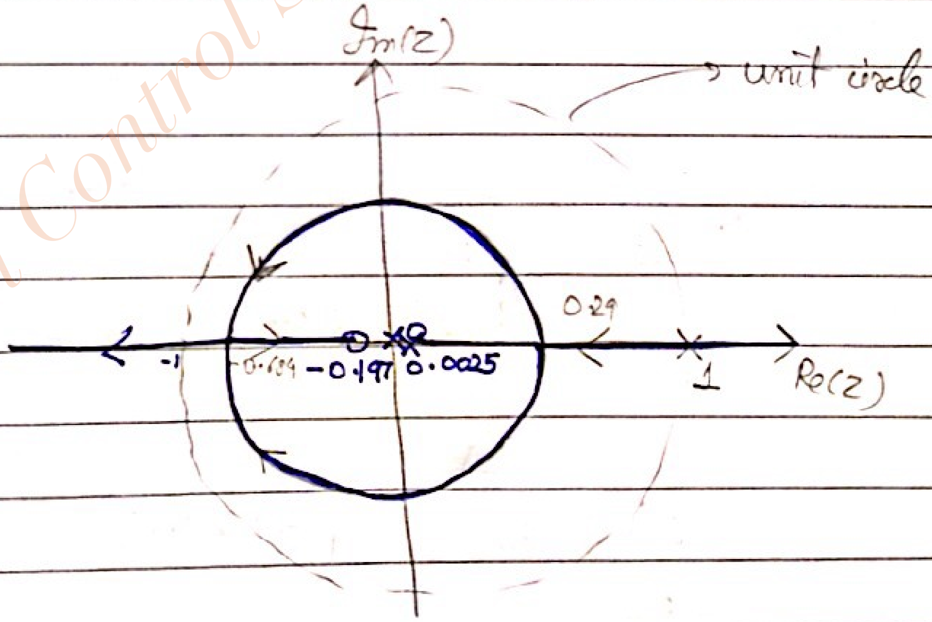


root locus circle centered at $z = -0.76$ rad $= 1.46$

root locus at $T = 0.45$
$K' = 0.75$

$K = 16K'$ $\quad K < 16 \times 0.75 = 12$

Taking $T = 3$ sec.



* Compensation by continuous network.

Consider CLTF for a discrete sys, unity f/b,

$$\frac{C(z)}{R(z)} = T(z) = \frac{Z[G_0(s)G(s)G_c(s)]}{1 + Z[G_0(s)G_c(s)G(s)]}$$

By block diagram represent^n, ZOH (holding) can be replaced by D(z)

So, TF:

$$\frac{C}{R}(z) = T(z) = \frac{D(z)\,Z[G_0(s)G(s)]}{1 + D(z)\,Z[G_0(s)G(s)]}$$ #

Now, finding steady state error,

$e(\infty)$ i.e error at $t \to \infty$

$\Rightarrow Z^{-1}(E(z))$

Now, $E(z) = R(z) - C(z)$. (Just like in cts domain, $E(s) = R(s) - C(s)$)

$$T(z) = \frac{C(z)}{R(z)}$$

$\Rightarrow E(z) = R(z) - T(z)R(z)$

$\Rightarrow E(z) = [1 - T(z)]R(z)$

By final value theorem,

$$e(\infty) = \lim_{z \to 1} (1 - z^{-1})E(z)$$

$$= \lim_{z \to 1} (1 - z^{-1})(1 - T(z))R(z)$$

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error coeff

Usually, we take inputs as:- $\delta \rightarrow$ impulse
$1 \rightarrow$ unit step $\rightarrow K_p$
$t \rightarrow$ ramp $\rightarrow K_v$
$t^2 \rightarrow$ accl^n $\rightarrow K_a$

So, in general, input can be

$$A(t)^q.$$

↳ Impulse : $t = q = 0$.
↳ unit step : $q = 0$
↳ ramp : $q = 1$
↳ accl^n : $q = 2$

Taking z transform of this i/p, say we get

$$R(z) = \frac{B(z)}{(1-z^{-1})^{q+1}}$$

$B(z)$ → any polynomial depends on $T(z)$
↳ Polynomial in z^{-1}.

So,

$$e(\infty) = \lim_{z \to 1} (1-z^{-1})(1-T(z)) \frac{B(z)}{(1-z^{-1})^{q+1}}$$

↳ $B(z)$ can have $(1-z^{-1})$ term. So,
$e(\infty) \rightarrow 0$.

$$\text{So,} \Rightarrow 1 - T(z) = (1-z^{-1})^{q+1}$$

$$\Rightarrow T(z) = 1 - (1-z^{-1})^{q+1}$$

Note, $E(z) = B(z)$: finite degree poly. in z^{-1}

Now, $D(z)$ can be got as

$$D(z) = \frac{1-(1-z^{-1})^{q+1}}{(1-z^{-1})^{q+1} \, Z[G_0(s)G(s)]}$$

* Finding E(z) at diff^t inputs :

① Step i/p. $(q=0)$

$$E(z) = (1-z^{-1})\frac{A}{(1-z^{-1})} = A.$$

$$Z^{-1}(E(z)) \quad , \quad e(0) = A \; ; \; k=0$$
$$e(kT) = 0 \; ; \; k = 1, 2, \ldots$$

② Ramp i/p $(q=1)$

$$E(z) = (1-z^{-1})^2 \frac{ATz^{-1}}{(1-z^{-1})^2} = ATz^{-1}$$

$$Z^{-1}(ATz^{-1}) \Rightarrow e(kT)\Big|_{k=0} = 0.$$
$$\Big|_{k=1} = AT$$
$$\Big|_{k=2,3,4\ldots} = 0$$

③ Accl^n i/p $(q=2)$

$$E(z) = (1-z^{-1})^3 \frac{AT^2 z^{-1}(1+z^{-1})}{(1-z^{-1})^3} = AT^2 \underbrace{z^{-1}}_{\delta(k-1)} + AT^2 \underbrace{z^{-2}}_{\delta(k-2)}$$

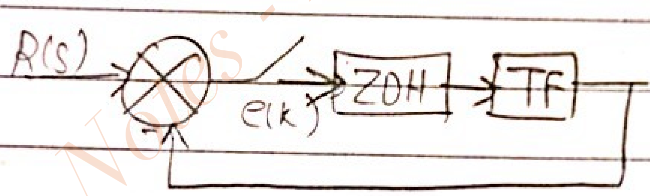
$$Z^{-1}\left(AT^2 z^{-1}(1+z^{-1})\right), \quad e(0) = 0$$
$$e(T) = AT^2$$
$$e(2T) = AT^2$$
$$e(kT)\Big|_{k=3,4,5\ldots} = 0.$$

★ Corresponding error coeff, K_p, K_v, K_a can also be found.

★ Frequency domain techniques for designing $D(z)$

eg: Consider a 2nd ord. sys. with TF: $\frac{k}{s(s+2)}$ alongwith sample & hold.

Design digital sys.
st:- $K_v \geq 4\ s^{-1}$
Phase margin $\geq 40^\circ$
BW ≤ 1 Hz (ie 2π rad/s)



① Selecting sampling freq
↳ using Nyquist criterion. $= 2 \times BW = 2$ Hz.
So, for good performance, $f_s \simeq 2.5$ Hz.
Corresponding $T = 0.4$ s.

For this control sys, we derived

$$G_0G(z) = \frac{k(z+0.76)}{16(z-1)(z-0.45)}$$

(with $T = 0.4$s thats why we chose $T = 0.4$s)

For K_v, find e_{ss} for ramp i/p

Now, Determining static gain

$$E(z) = R(z) - C(z) = \frac{R(z)}{1+G_0G(z)}$$

Using final value theorem,

$$e_{ss} = \lim_{k\to\infty} e(kT) = \lim_{z\to 1} (1 - z^{-1}) E(z)$$

Now, for ramp i/p, kT,

$$R(z) = T\left(\frac{z}{(z-1)^2}\right)$$

So,

$$e_{ss} = \lim_{z\to 1} \frac{T}{(z-1)(1+G_0G(z))} = \frac{1}{K_v}.$$

$$\text{So, } K_v = \frac{1}{T} \lim_{z\to 1} (z-1)\underbrace{\left[G_0\, G(z)\right]}_{\text{given}}$$

So, we get $K_v = \frac{K}{2} \geq 4$ → given K_v should be ≥ 4

So, $K \geq 8$

So, take $K = 8$.

Now, considering condⁿ: phase margin $\geq 40°$
That is done/seen using bode plot. (implemented using MATLAB)
For using this,
Now, deriving cts domain TF using BZT

My open loop sys, TF $= G_0G(z) = \dfrac{0.5(z+0.76)}{(z-1)(z-0.45)}$.

Use : The transformⁿ : $z = \frac{1+h}{1-h}$

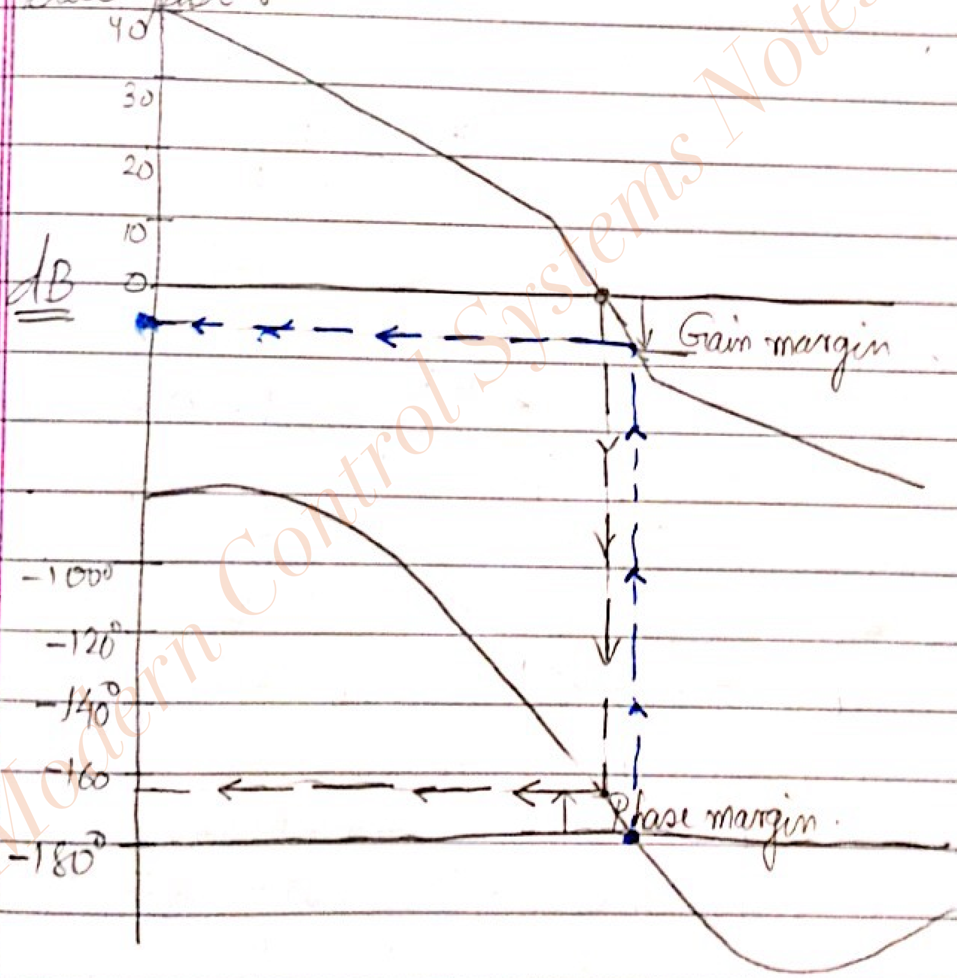
So, $G_0G(h) = \frac{(0.8)(-h+1)(h/7.35+1)}{h(h/0.38+1)}$

For freq. in h-plane,

$h \rightarrow j\omega_h$

$\Rightarrow G_0G(j\omega_h) = \frac{(0.8)(-j\omega_h+1)(j\omega_h/7.35+1)}{(j\omega_h)(j\omega_h/0.38+1)}$

Bode plot :



※ Design of compensation :- $D(s) = \frac{1+st}{1+\beta st}$ or $D(h) = \frac{1+Th}{1+\beta hT}$

$\beta \geq 1$: lag

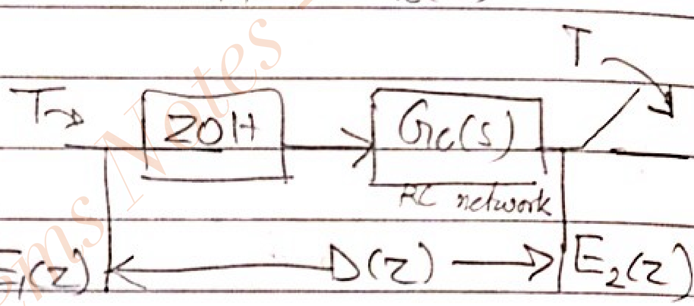
$$D(\lambda) = \frac{1 + 23.2\lambda}{1 + 88.8\lambda}$$

Transformⁿ to Z-domain : $\lambda \to \frac{z-1}{z+1}$

$\therefore D(z) = 0.258\left(\frac{z - 0.915}{z - 0.978}\right)$: Compensated TF

Now, Implementⁿ of TF :

Use of RC network :
TF = $G_c(s)$



So, $D(z) = \mathcal{Z}\left[\left(\frac{1 - e^{-Ts}}{s}\right) G_c(s)\right] = 0.258\frac{(z - 0.915)}{(z - 0.978)}$

So, $\mathcal{Z}\left\{\frac{G_c(s)}{s}\right\} = \frac{D(z)}{1 - z^{-1}} = \frac{0.258(1 - 0.915z^{-1})}{(1 - z^{-1})(1 - 0.978z^{-1})}$

$$= \underbrace{\frac{1}{1 - z^{-1}} - \frac{0.775}{1 - 0.978z^{-1}}}$$

Implementⁿ using Direct form 1 or direct form 2 (DSP)

So, $D(z) = \frac{E_2(z)}{E_1(z)} = 0.258 \frac{(1-0.915z^{-1})}{(1-0.978z^{-1})}$

$\Rightarrow (1-0.978z^{-1})E_2(z) = 0.258\,(1-0.915z^{-1})E_1(z)$

So,

$e_2(k) = 0.258e_1(k) - 0.246e_1(k-1) + 0.978e_2(k-1)$

↳ can be implemented in computer.

CLOSED LOOP FREQUENCY RESPONSE

Using freq. domain techniques to implement time domain specifications.

- frequency responce ⇒ transient response ⇒ time domain specs.
- compensators designed in freq. domain.

* Frequency domain specs :

1. Resonance peak M_r : max. value of CL mag. response
2. Resonant freq, ω_r : freq at which we get M_r.
3. Bandwidth : Sys. gain > −3dB (freq. range)
   indicates the ability of sys. to reproduce i/p signal and a measure of noise rejection characteristics
4. Cut off rate : Slope of log(mag. curve) near cut off freq
   ⇔ ability to distinguish the signal & noise.
   → tells how fast transⁿ takes place from PB & SB.

5 Gain margin & phase margin : Measure of relative stability ⇒ indicates closeness of CL poles to $j\omega$ axis i.e, how much change in gain / phase angle can be done before sys. becomes unstable

- Important parameters are M_r & ω_r.
  analytical method is difficult. So, using

☆ Graphical method

↳ to find M_r & ω_r using constant M & N circles.

↳ constant M circle

let $G(j\omega) = x + jy$ on polar plot

The CL freq response is

$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)} = \frac{x+jy}{(1+x)+jy}$$

$$\Rightarrow Me^{j\alpha} \; ; \; M = \frac{|x+jy|}{|1+x+jy|} \Rightarrow M^2 = \frac{x^2+y^2}{(1+x)^2+y^2}$$

Rearranging,

$$\Rightarrow y^2 + \left[x + \frac{M^2}{M^2-1}\right]^2 = \frac{M^2}{(M^2-1)^2}$$

$$\text{///}$$

$$(x^2 + y^2 = r^2)$$

↳ eq^n of circle with center : $\left(\frac{-M^2}{M^2-1}, 0\right)$

radius $\left(\frac{M}{M^2-1}\right)$

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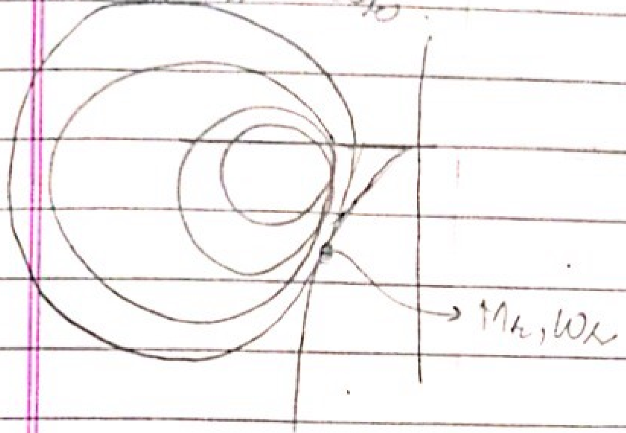
For OLTF, we can make a polar plot for $G(j\omega)$. Now, make const M circles on the same plot. Now, One of the circles will be tangent to polar plot of $G(j\omega)$. The tangent point is M_r. Corresponding ω is ω_r.



↳ constant N circles

Phase angle, $T(j\omega)$

$$\angle T(j\omega) = \alpha = \angle\left(\frac{x+jy}{1+x+jy}\right)$$

$$= \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{1+x}\right)$$

$$= \tan^{-1}\left(\frac{y}{x^2+x+y^2}\right)$$

$$\Rightarrow \tan\alpha = \frac{y}{x^2+x+y^2} = N$$

↳ if α = constt, $\tan\alpha$ = constt

$\Rightarrow N$ = constt.

Rearranging & adding both sides by $\frac{N^2+1}{4N^2}$

$$\Rightarrow \left(x+\tfrac{1}{2}\right)^2 + \left(y-\tfrac{1}{2N}\right)^2 = \frac{N^2+1}{4N^2}$$

$$(x^2 + y^2 = r^2)$$

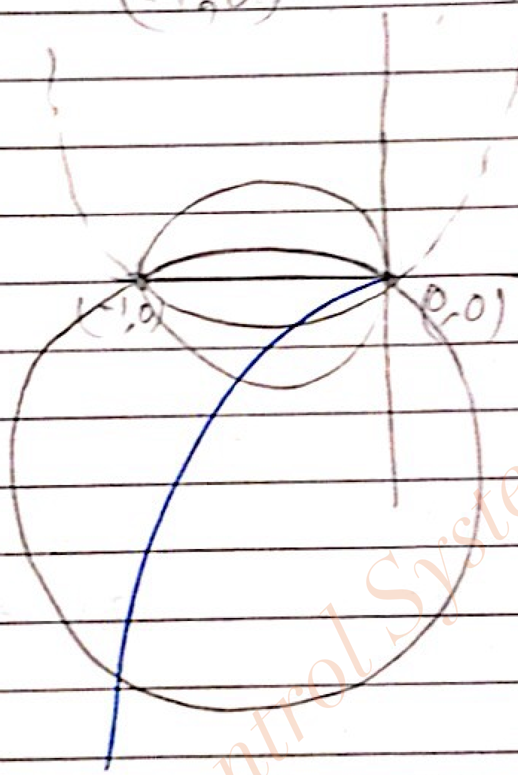
freq. range $-\infty$ to ∞ : Polar Plot
0 to ∞ : Nyquist Plot

Again acts as a circle

Center :- $\left(-\frac{1}{2}, \frac{1}{2N}\right)$

radius :- $\sqrt{\frac{N^2+1}{2N}}$

So, constt N circles pass through origin & $(-1,0)$



Intersection of N circle at M_r, ω_r point gives phase corresponding to that point.

Q. Given OLTF, $G(j\omega) = \dfrac{10}{j\omega(1+0.2j\omega)(1+0.05j\omega)}$ → Open loop gain (Got in Time constt form)

(time constt form)

Find :- CL Freq response of given OLTF TF using constt M & constt N circles.

Now, converting to Polar form :

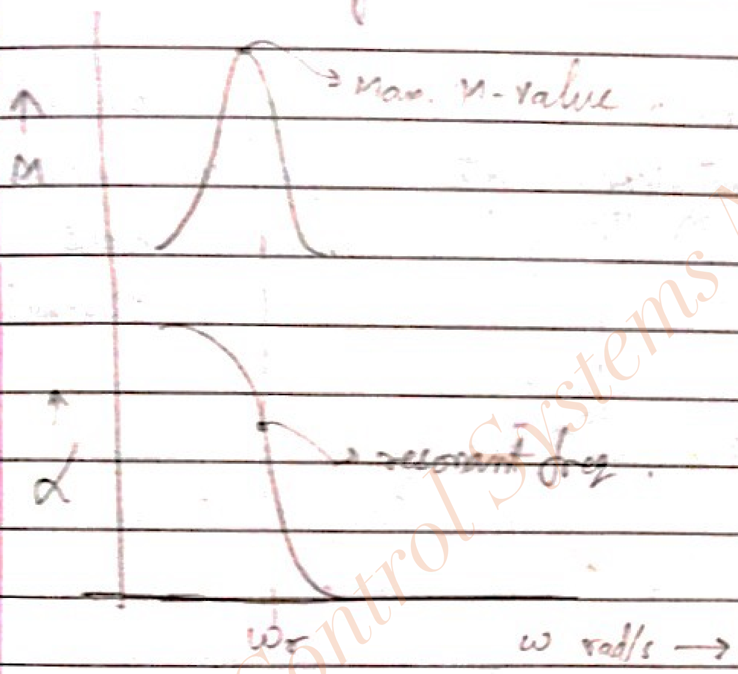
$$G(s) = \frac{10}{(0.2)(0.05)(s)(s+5)(s+20)}$$

→ Magnitude plot

(iii) It'll reach from ∞ to 0 along $-270°$ axis (when $\omega \to \infty$)

Plotting Nyquist plot for each of the poles for given OLTF

		Intersection of nyq plot with:
Magnitude of closed loop sys.	:	M-circle
Phase angle	:	N-circle



✧ Considering Non-Unity feedback sys :-

i.e., now, I have : $\dfrac{C(j\omega)}{R(j\omega)} = \dfrac{G(j\omega)}{1+G(j\omega)H(j\omega)}$

let $T(j\omega) = \dfrac{1}{H(j\omega)}\left[\dfrac{G(j\omega)H(j\omega)}{1+G(j\omega)H(j\omega)}\right]$

$= \dfrac{1}{H(j\omega)}\left[\dfrac{G_0(j\omega)}{1+G_0(j\omega)}\right]$

$= \dfrac{1}{H(j\omega)} \times T_0(j\omega)$

where, $T_0(j\omega) = \dfrac{G_0(j\omega)}{1+G_0(j\omega)}$, say

* In bode plot, on knowing graph of $T_0(j\omega)$ & $H(j\omega)$, we get $T(j\omega)\left(= \dfrac{T_0(j\omega)}{H(j\omega)}\right)$ by subtraction.

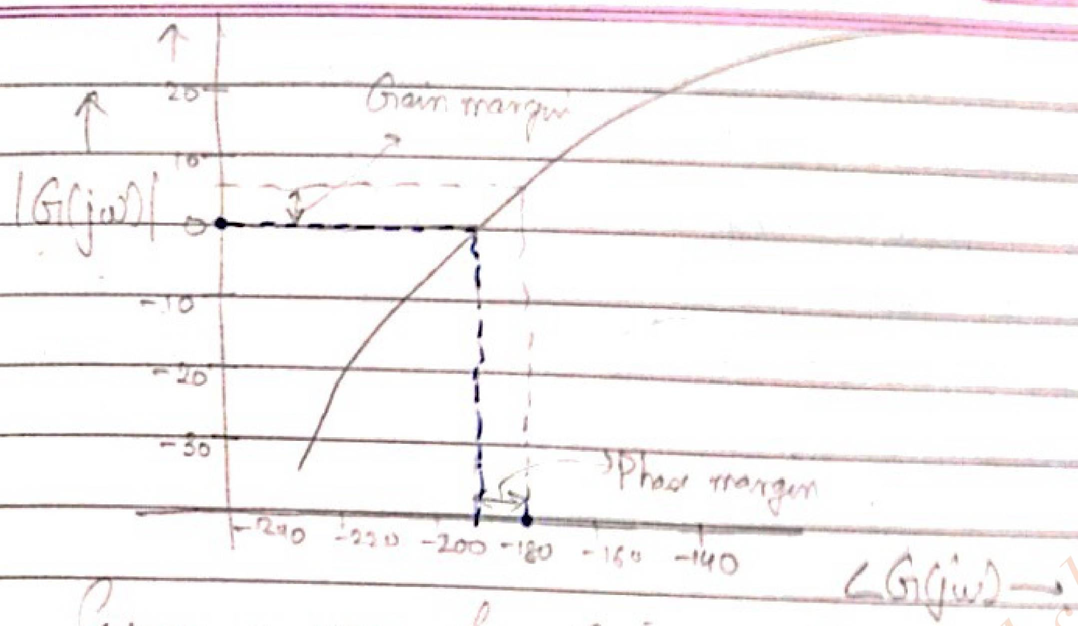
* Constt M & constt N contours on polar plot are circles.
  Constt M & N circles to log magnitude & phase angle coordinate (together imposed on same graph) is called Nichols chart.
  - ↳ useful to determine CL freq. response from that of open loop.
  - ↳ Gain Adjustments can be done.

eg Given:- OLTF:

$$G(j\omega) = \frac{10}{(j\omega)(j0.1\omega+1)(j0.05\omega+1)}$$

After getting magnitude & phase angle from constt M-circle & N-circle plot, plot Gain margin Vs phase margin.

Find: Given OLTF, find values of OL gain k, so that Gain margin = 20 dB & phase margin = 24°.



Suppose a case when Gain margin is as 12 dB, we wanted 20 dB. So, make graph 8 dB down.
Illy, move graph for phase margin.

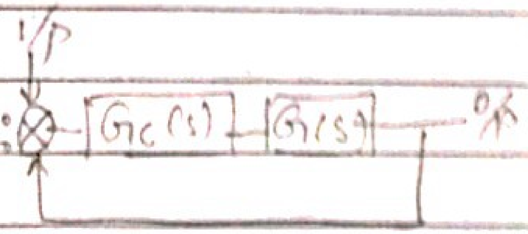
★ A good control sys. will adjust itself wrt change in sys. parameters
So, seeing.

SENSITIVITY in freq. domain,

$$S_{K}^{T} = \frac{\partial T / T}{\partial K / K}$$

K → OL gain.

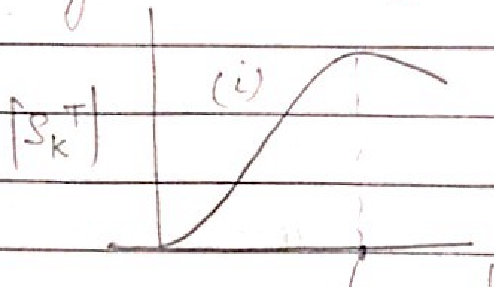
eg: consider a TF with sys:



So, $T(s) = \frac{25K}{s^2 + 5s + 25K}$

$$S_{K}^{T(s)} = \frac{s(s+5)}{s^2+5s+25} = \frac{\partial T}{\partial K}$$

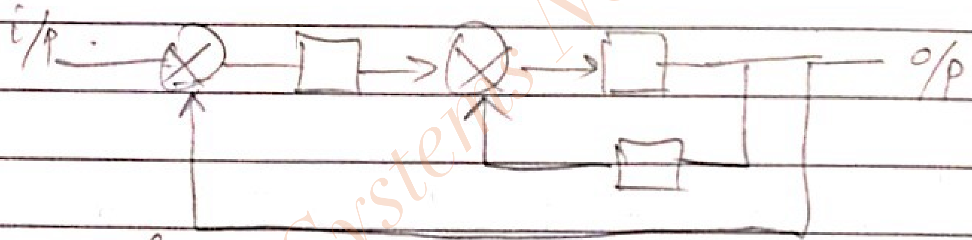
Now, $s \rightarrow j\omega$.
Magnitude of fn w.r.t change in freq. (ω) gives sensitivity:



We are seeing at what value does the K value affect the sys. the most

Pt. of max. sensitivity

(ii) Now, consider a double loop sys. → like:

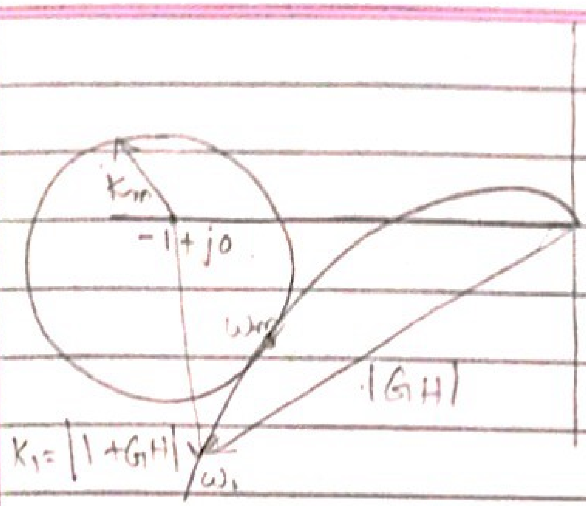


we find that the sensitivity graph is lower than for single loop.

$$S_K^T(j\omega) = \frac{j\omega(j\omega+1)}{(j\omega)^2 + 5(j\omega) + 25}$$

☆ Sensitivity of CLTF (T) to variations in OLTF (G).

$$S_G^T(j\omega) = \frac{1}{1 + G(j\omega)H(j\omega)} = \frac{1}{K_1}\text{, say}$$



* Max. sensitivity is corresponding to K_m circle. (i.e M circle with radius K_m) In this case K_m should be min. (for max. sensitivity)

✓ Corresponding freq.: ω_m

☆ State variable & linear discrete time systems :

General form :

$$[x(k+1)T] = f[x(KT), u(KT)]$$

$$y(KT) = g[x(KT), u(KT)]$$

In discrete time sys.

Derivative = difference in values

So, * $\dot{x}(k) = x(k+1)$

So, $\dot{x} = Ax + Bu$

$\equiv$ $x(k+1) = Ax + Bu$

$= f(x, u)$

& $y = Cx + Du$

$= f(x, u)$

$$\therefore \quad x(k+1) = Ax(k) + Bu(k) \quad ; x(KT) \triangleq x(k)$$

$$y(k) = Cx(k) + Du(k)$$

$x(k) = n\times 1$ state vector
$u(k) = m\times 1$ i/p vector
$y(k) = p\times 1$ o/p vector
$A = n\times n$ sys matrix
$B = n\times m$ i/p matrix
$C = p\times n$ o/p matrix
$D = p\times m$ transmission matrix (feed forward matrix)

- **State models from Linear Difference Eq^ns or z-transfer f^ns.**

Consider a 3rd ord. TF :-

$$T(z) = \frac{Y(z)}{U(z)} = \frac{b_0 z^3 + b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}$$

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

$a_0 = 1$ ∀ unity f/b sys.

FORM I PHASE VARIABLE FORM

State space represent^n

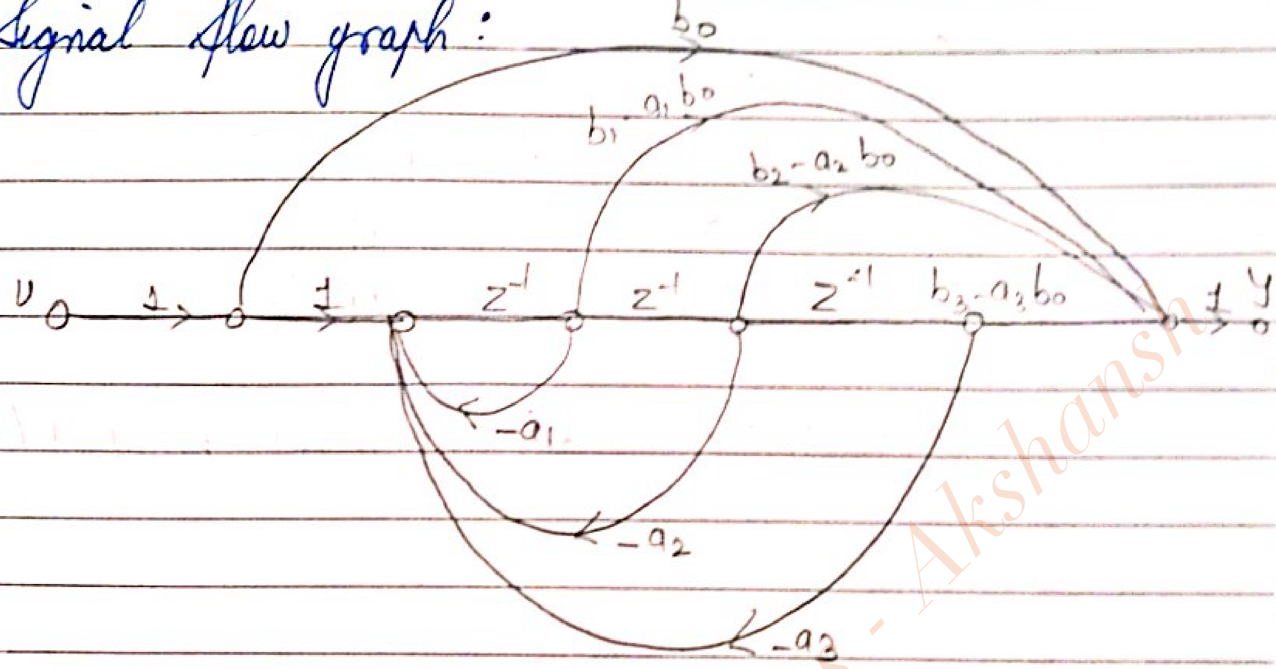
(let $x_1, x_2, x_3 \rightarrow$ state vars)

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = x_3(k)$$
$$x_3(k+1) = -a_3 x_1(k) - a_2 x_2(k) - a_1 x_3(k) + u(k)$$

&

$$y(k) = (b_3 - a_3 b_0) x_1(k) + (b_2 - a_2 b_0) x_2(k) + (b_1 - a_1 b_0) x_3(k) + b_0 u(k)$$

Signal flow graph:



State space model :-

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} (b_3 - a_3 b_0) & (b_2 - a_2 b_0) & (b_1 - a_1 b_0) \end{bmatrix}$$

FORM 2 CANNONICAL FORM

Let $T(z) = \dfrac{z^3 + 8z^2 + 17z + 8}{z^3 + 6z^2 + 11z + 6}$

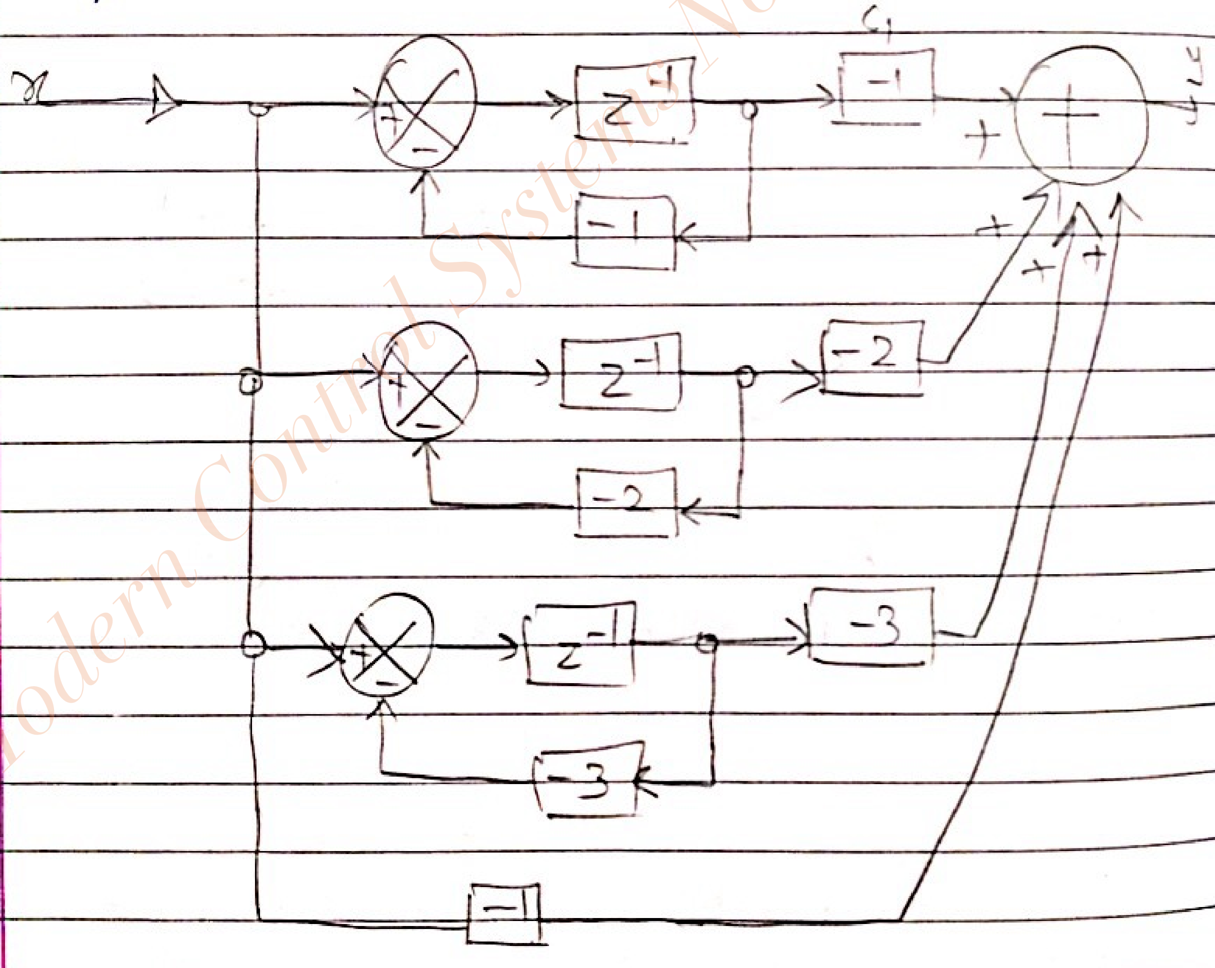
$$= 1 - \frac{1}{z+1} + \frac{2}{z+2} + \frac{1}{z+3}$$

Diagonal form :

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + u(k)$$

Implementation :



z-transform from discrete time state model

Taking z-transform :

$$zX(z) - z x(0) = AX(z) + BU(z)$$

Solving for $X(z)$

$$\Rightarrow X(z) = (zI-A)^{-1} z x(0) + (zI-A)^{-1} BU(z)$$

$$Y(z) = CX(z) + dU(z)$$

$$\Rightarrow y(z) = C(zI-A)^{-1} z x(0) + C(zI-A)^{-1} BU(z) + dU(z)$$

let initial condⁿˢ = 0

$$\Rightarrow T(z) = \frac{Y(z)}{U(z)} = C(zI-A)^{-1}B + d$$

$$\Rightarrow T(z) = C\left(\frac{\operatorname{adj}(zI-A)}{|zI-A|}\right)B + d$$

Char. eqⁿ $\Rightarrow |zI-A| = 0$

Now, for time domain form : finding

State transⁿ matrix or Resolvent matrix
(trajectory of state)

✓ **State eqⁿˢ (Discrete case)**

here, we will have samples at each instants : for $x(k) \to x(1), x(2) ---$

$$x(1) = Ax(0) + Bu(0)$$

$$x(2) = Ax(1) + Bu(1) = A^2x(0) + ABu(0) + Bu(1)$$

$$\vdots$$

$$x(k) = A^k x(0) + A^{k-1}Bu(0) + A^{k-2}Bu(1) + \dots Bu(k-1)$$

Now, defining :-

$$\phi(k) \triangleq A^k$$

So, $\phi(0) \triangleq I$ (Identity matrix)

So, we get

$$\boxed{x(k) = \phi(k)\, x(0) + \sum_{i=0}^{k-1} \phi(k-i-1)\, B u(i)}$$

Initial cond^n = 0, if not given

Now,

Observability & Controllability : Discrete Domain.

↳ here we have "T" as an additional parameter. So, apart from other cond^{n}s, ∃ one more cond^n :

$$* \boxed{T \neq \frac{2\pi n}{\omega}}$$

↳ T : sampling interval.

↳ ∵ TF in z-domain becomes something like

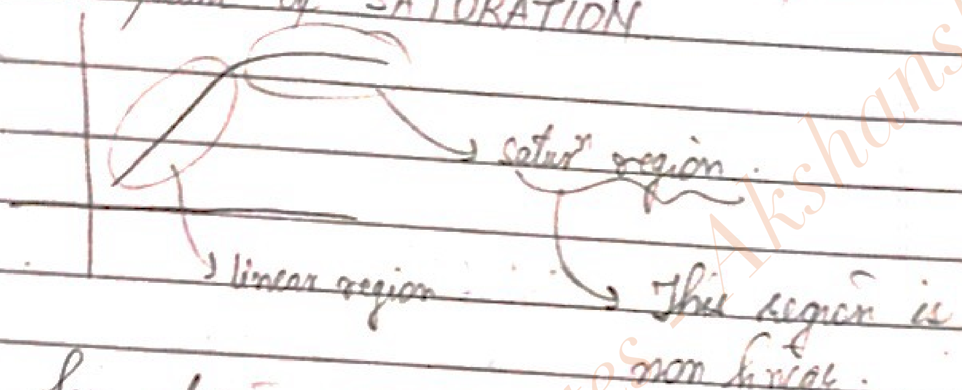
$$\frac{z^{-1} e^{-\sigma T} \sin \beta T}{1 + (\ ) + (\ ) + \ldots}$$

$= 0$, $T = \frac{2n\pi}{\omega}$

So, $G(z) = 0$. So, sys becomes non observable or non-controllable. So, this cond^n is reqd.

§ NON-LINEAR SYSTEM STABILITY

In any sys. that we consider, a lot of times, we get a point of SATURATION



Now, for design purpose, we are trying to see how to incorporate that region.

* LIAPUNOV'S Stability
  (valid ∀ linear & non linear sys)
  ↳ Defin^ns :
  - Autonomous sys : $\dot{x} = F(x)$
  - Singular pt :- Pts. in the phase-plane at which derivatives of all state variables are zero.
    (derivative = 0 ⇒ slope = 0 ⇒ sys is in equilibrium)
  - Sys. Stability for : ① free system
    ② forced system

for linear sys.
① free sys :- The sys. in state with zero i/p & arbitrary initial cond^ns if the resulting trajectory tends to equilibrium state
② forced sys : Sys. is stable for BIBO (Bounded i/p bounded o/p)

For non linear sys, ∃ no definite correspondence b/w criterion ① & ②.

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Sys. behaviour for small deviation about equilibrium pt. is diff^t from large deviation for non linear sys, we define stability wrt deviations from equilibrium pt :
i.e local stability & overall state plane stability are diff^t :
i.e for non linear sys$_s$ here
1) Stability
2) Asymptotic stability
3) Asymptotic stability in large

1) Normal Stability
$x = F(x)$ is stable at origin
$\forall\ x(t_0)$, $x(t)$ remains near origin $\forall\ t$

→ Sys. will come back to original pos^n even if initial cond^{ns} is varied largely.

2) Asymptotic stability
$x(t) \to$ origin as $t \to \infty$.

3) Asymptotic stability in large
If its asymptotically stable for every initial cond^n.

Now, consider a sys :
$\dot{x} = f(x, v)$
& $\dot{x} = F(x)$
$= Ax$
As $A \neq 0$, so, for equilibrium cond^n to be true, $x = 0$. So, equilibrium state is at origin
So, $x = x_e = 0$
x_e → equi
→ x_e is a singular pt.

If x_e is moved or origin is moved to x_e, new phase variables

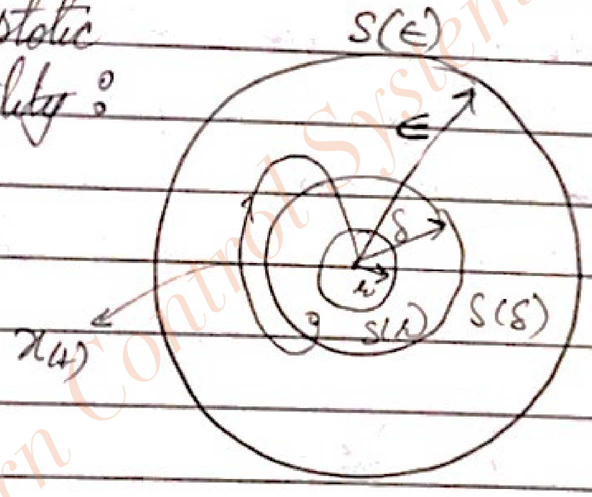
$$\tilde{x} = x - x_e \quad \&$$
$$\dot{\tilde{x}} = F(\tilde{x}) \quad \&$$
$$x_e = \tilde{x}.$$

Consider, $S(R)$: Area surrounding equilibrium pt with radius R ($\tilde{x} = 0$)

Stability at Origin :

(assigning max. value of radius for stability)

Asymptotic Stability :



(assigning max. & min value of radius for equilibrium s.t sys is stable)

— e —

* Limit cycle :

Seen in → Vander Pol's differential eq^n :-

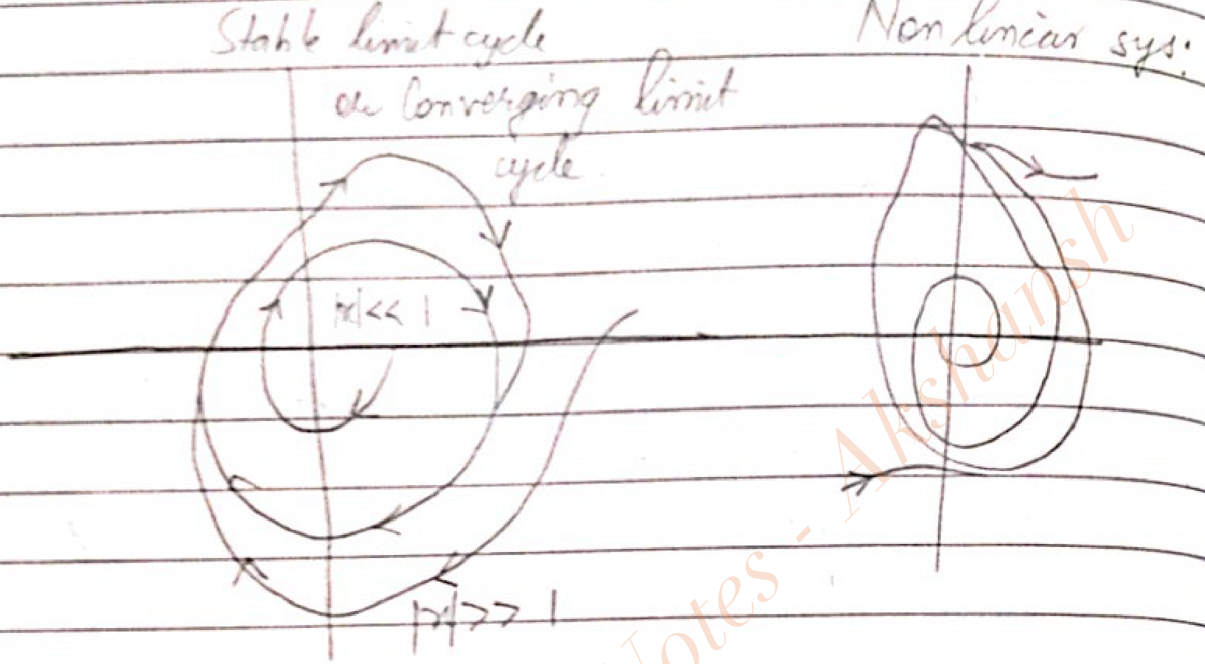
(non linear sys) $$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

(linear sys) $$\frac{d^2x}{dt^2} + (2\zeta)\frac{dx}{dt} + x = 0$$

x = state variable

Cases:

If $x >> 1$, $(1-x^2) \longrightarrow$



- In linear sys autonomous signals, when ~~or~~ oscill^n occurs, resulting trajectory is called curves
- Amplitude of oscillator is not fixed but dependent on size of ~~order~~ initial cond^n.
- Slight changes in sys parameters ~~dest~~ destroy the oscill^n.

for non linear eq^n:-

Liapunov's Stability criterion

$$\dot{x} = f(x(t), u(t)) \; ; \; \dot{x}(t) = f(x(t))$$

let

$x(x(t_0), t)$ be a sol^n.

$V(x)$ – total energy associated with sys.; if

$\frac{d}{dt} V(x) < 0 \; \forall \; x(x(t_0), t)$ except at equilibrium pt $\Rightarrow$ energy $\downarrow$ as $t \uparrow \rightarrow 0$ at equilibrium eg:- Consider state eqns for a mechanical sys:-

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -Kx_1 - fx_2$$

Total energy, $V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{2}Kx_1^2$

→ $V(x) > 0$ when $x \neq 0$
→ $V(0) = 0$, $x = 0$

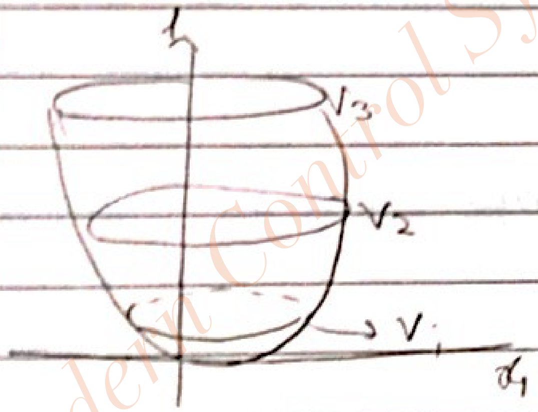
Now,

$$\frac{d}{dt}V(x_1, x_2) = \frac{\partial V}{\partial x_1} \cdot \underbrace{\frac{dx_1}{dt}}_{\dot{x}_1} + \frac{\partial V}{\partial x^2} \cdot \underbrace{\frac{dx_2}{dt}}_{\dot{x}_2}$$

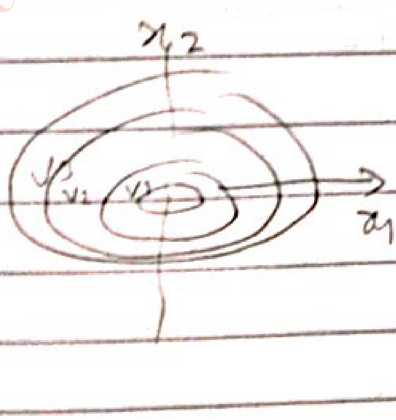
Making partial fractions :- & substituting ⇒

$$\frac{d}{dt}V(x_1, x_2) = Kx_1 \cdot x_2 + x_2(-Kx_1 - fx_2)$$
$$= -x_2^2 f_2$$

(sys. reaches 0 at equilibrium)



We find that boundry pts are oval shapped.



→ Lyapunov associated :- The energy f^n with the given sys. The scalar used to define the stability, is called LIAPUNOV's f^n

~~Liapunov~~

❋ LIAPUNOV'S THEOREM

Theorem 1: (Normal Stability Check)

Let $\dot{x} = f(x)$; $f(0) = 0$

Suppose $\exists$ a scalar $V(x)$ for some real $\epsilon > 0$ satisfies following property $\forall\ x \Rightarrow \|x\| \leq \epsilon$.

a) $V(x) > 0$; $x \neq 0$

↳ +ve definite scalar f^n

b) $V(0) = 0$ i.e $V(x) = 0$; $x = 0$

c) $V(x)$ has continuous partial derivatives wrt all components of x.

d) $\dfrac{dV}{dt} \leq 0$ (i.e, $\dfrac{dV}{dt}$ is NSD Scalar f^n)

↳ −ve semi definite

$(<)\ (=)$

If a, b, c & d are satisfied ⇒ sys. is stable.

(Normal Stability)

Theorem 2: (Asymptotic stability in small)

(a), (b) & (c) are same as Theorem ①.

(d) $\dfrac{dV}{dt} < 0$; $x \neq 0$

(i.e, $\dfrac{dV}{dt}$ is NDS f^n)

In this case, sys. is asymptotically stable

Theorem 3: Alongwith all cond^ns in theorem (2)
Asymptotic Stability in large

(e) $V(x) \to \infty$ as $\|x\| \to \infty$.

Then, sys. is asymptotically stable in large at origin

★ LIAPUNOV FUNCTION

→ finds scalar qty, $V(x)$.

The determin^n of stability via Liapunov direct method depends on the choices of a PD f^n, $V(x)$ called Liapunov f^n.

$V(x)$ (in quadratic form) can be determined using

Sylvester's Theorem:

Instability can be defined & stated in Theorem (4):

Theorem 4: Consider sys: $\dot{x} = f(x)$; $f(0) = 0$
Instability
for all x s.t $\|x\| \leq \epsilon$, following cond^ns should be satisfied:

(a) $W(x) > 0$; $x \neq 0$.

(b) $W(0) = 0$

(c) $W(x)$ has cts partial derivatives wrt all components of x

(d) $\frac{dW}{dt} \geq 0$.

13.1

Q. Consider non linear sys governed by :-

$$\dot{x}_1 = -x_1 + 2x_1^2 x_2$$
$$\dot{x}_2 = -x_2$$

Now,

Choosing V (seeing the first 3 condⁿˢ of previously written theorems)

$$V = x_1^2 + x_2^2$$ (Liapunov fn ↳ will be given in exam)

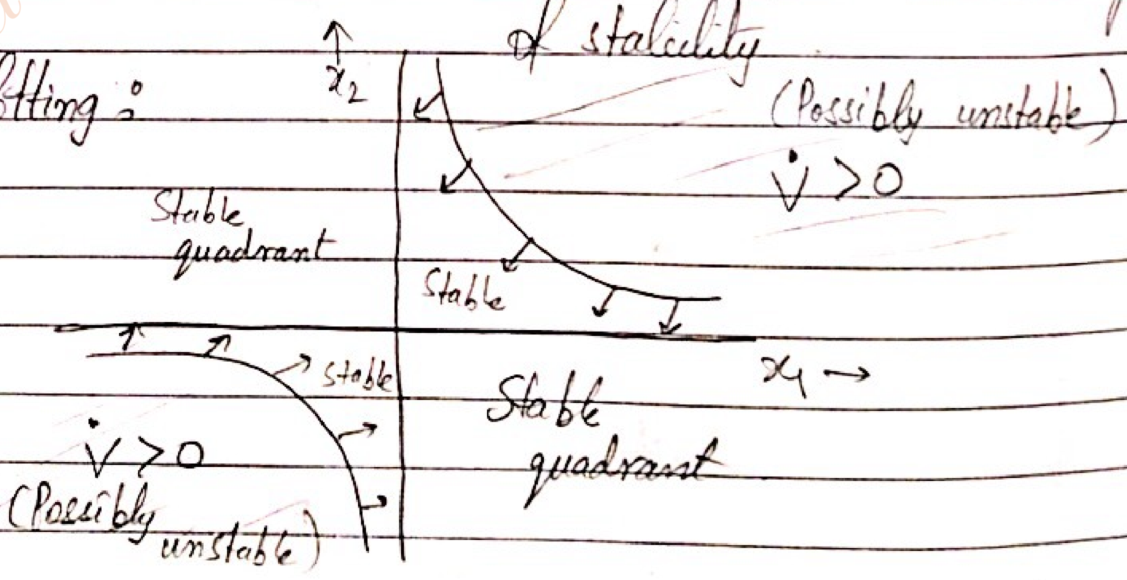
Using 3rd condⁿ of theorem,

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\cdot\frac{dx_2}{dt}$$
$$= -2x_1^2(1-2x_1x_2) - 2x_2^2$$

→ here, if $1 - 2x_1x_2 > 0$, the sys. is -ve definite. So, it's Asymptotic Stability in large

→ for $1-2x_1x_2 < 0$, we will have to see regions of stability

Plotting :



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Q, Consider another non linear sys:-

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - x_1^3.$$

Clearly, origin is equilibrium pt

Taking Liapunov's fn:-

$$V = x_1^4 + (x_1 + x_2)^2$$

(mostly, will be given)

Derivative of Liapunov fn:-

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

$$= -2x_1^4 - 2x_2^2$$

#

☆ ~~for~~ Liapunov Stability applied to Linear Systems:

Consider a linear sys, $\dot{x} = Ax$.

Linear sys is asymptotically stable in Large iff:- given +ve ~~symmetric~~ definite matrix Q,

∃ symm. +ve definite matrix P which is unique soln

"$A^T P + PA = -Q$."

Idea : find matrix P & check if it satisfies

$A^T P + PA = -Q$.

If its +ve definite, sys is stable.

Proof :-

Suppose P is symmetric definite +ve matrix (assume)

Consider, scalar f^n,

$$V(x) = x^T P x$$

$$V(x) > 0, \quad x \neq 0 \longrightarrow (1)$$

$$V(0) = 0 \longrightarrow (2)$$

Time derivative of $V(x)$,

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$$

Using these eq^{ns} :-

$$\dot{V}(x) = x^T A^T P x + x^T P A x$$

$$= x^T (A^T P + PA) x$$

$$= -x^T Q x \longrightarrow (3)$$

For stability to be satisfied,

$$\dot{V}(x) < 0.$$

So, $Q > 0$ (reverse verific^n of assumption)

Also, $\|x\| = (x^T P x)^{1/2}$ (by defin^n of NORM f^n)

f^n NORM of x

Then, $V(x) = \|x\|^2$

$\Rightarrow V(x) \to \infty$ as $\|x\| \to \infty$

(So, asymptotically stable in large)

Note : Derivative +ve $\Rightarrow$ Unstable

Q Determine stability of sys given by

$$\dot{x} = Ax$$

$$\hookrightarrow A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}$$

Assuming as Identity matrix for Q

$$Q = I$$

So, eq^n becomes

$$A^T P + PA = -I$$

i.e

$$\begin{bmatrix} -1 & 1 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\hookrightarrow$ +ve symmetric

i.e $P_{12} = P_{12}$

Writing eq^n for each row, equating elements.

$$\Rightarrow -2P_{11} + 2P_{12} = -1$$

$$-2P_{11} - 5P_{12} + P_{22} = 0$$

$$-4P_{12} - 8P_{22} = -1$$

Solving P, we get

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} 23/60 & -7/60 \\ -7/60 & 11/60 \end{bmatrix}$$

Using Sylvester's theorem,

$\hookrightarrow$ necessary & sufficient cond^n for a matrix to be +ve definite :

All minors (principal) of $Q > 0$.

i.e $q_{11} > 0$, $\begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix} > 0$, $\begin{vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix} > 0$

So, basically,

$$\det[Q] > 0$$

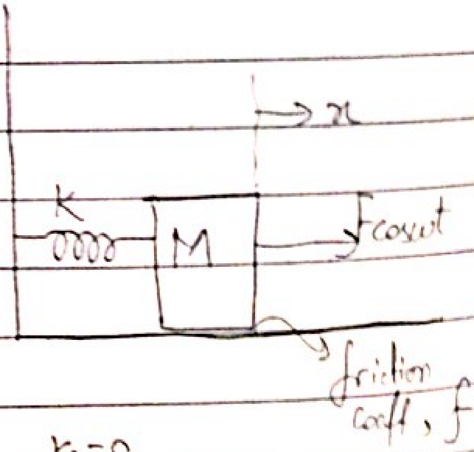
(Q is semidefinite if any of minors is "0")

Non linear Systems

↳ don't follow POS & homogeneity
↳ may exhibit limit cycles
↳ self sustained oscil^n with fixed amplitude & freq.

Consider a linear fn:-

$$M\ddot{x} + f\dot{x} + kx = F\cos\omega t$$



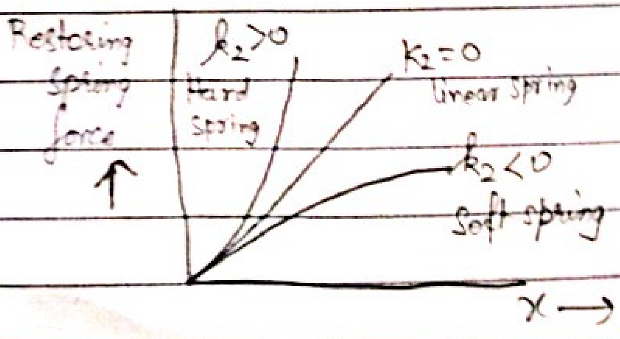
Assuming restoring force of spring is non linear

$$\Rightarrow k_1 x + k_2 x^3$$

Linear, if $k_2 = 0$

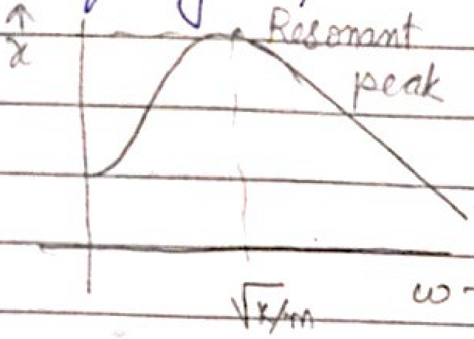
Hard , $k_2 > 0$

Soft , $k_2 < 0$
spring

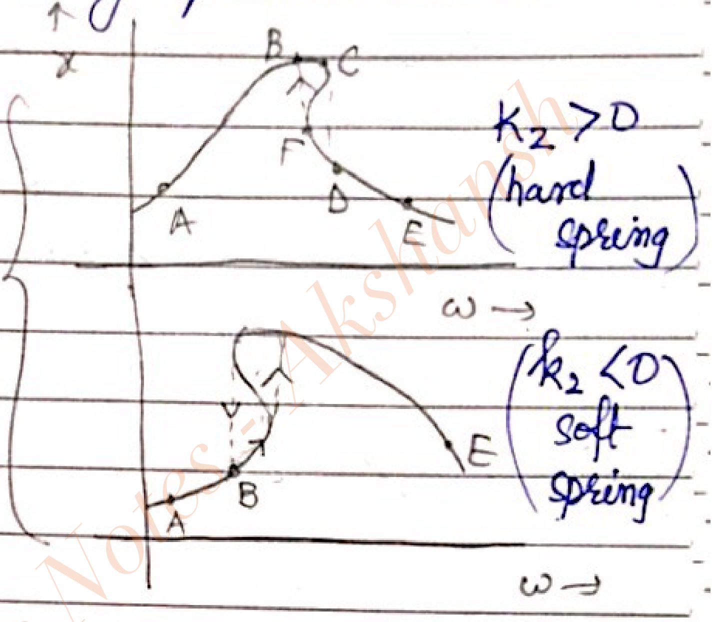


i.e, Non linear spring is : $M\ddot{x} + f\dot{x} + K_1 x + K_2 x^3 = F\cos\omega t$

Frequency response



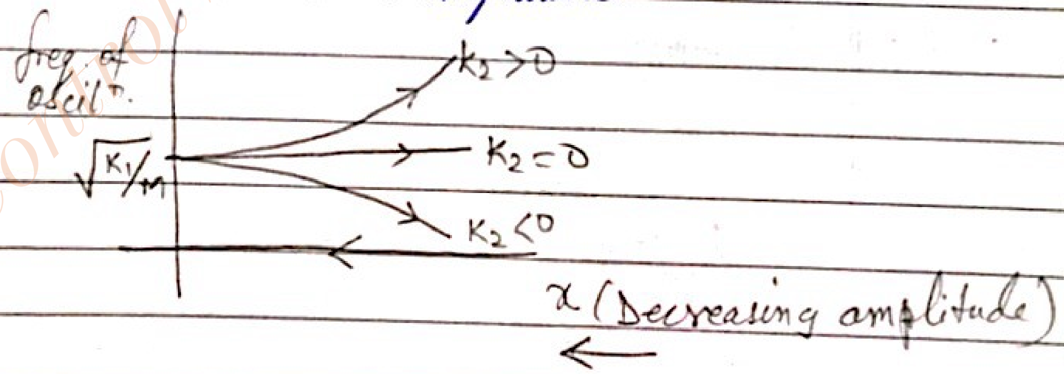
Jump resonance :



✓ Corresponding to 1 i/p we have 2 o/ps.

✓ we see discontinuities in responses.

We find :- In non linear sys, freq. response is sensitive to amplitude.



COMMON PHYSICAL NON-LINEARITIES :

1) Inherent or Incidental non linearity.

↳ inherent in sys

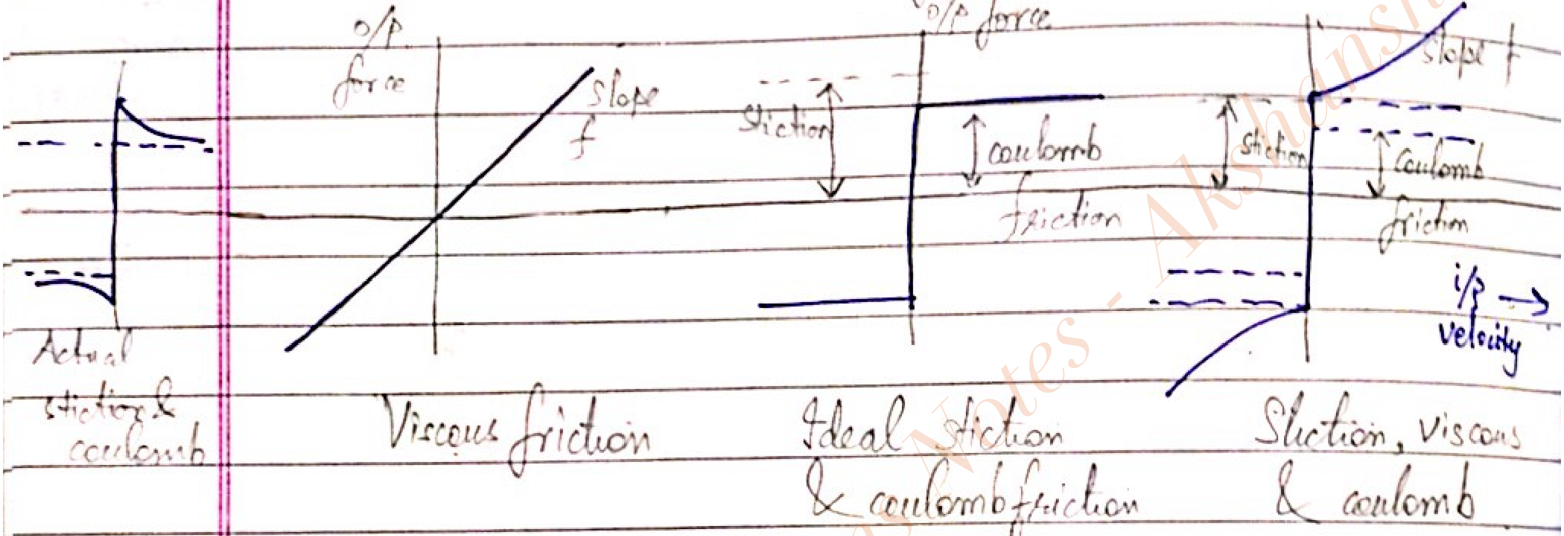
eg: satur^n, coulomb friction etc.

2) Intentional non linearity

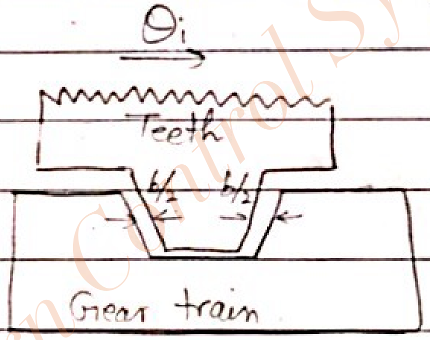
↳ deliberately included.

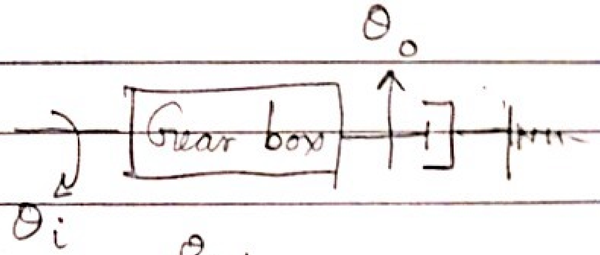
eg: relays.

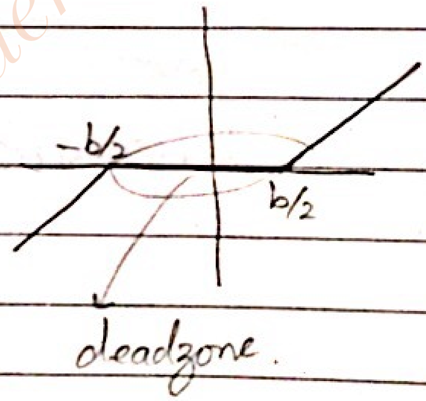
* Friction :
  > Viscous friction : Linear ⇒ $F = f\dot{x}$
  > Coulomb friction : constt. retarding force (always opposing)
  > Stiction : force to initiate motion (always > coulomb friction)

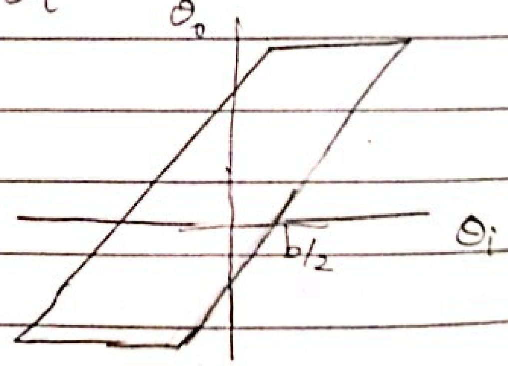


* Backlash Non linearity :



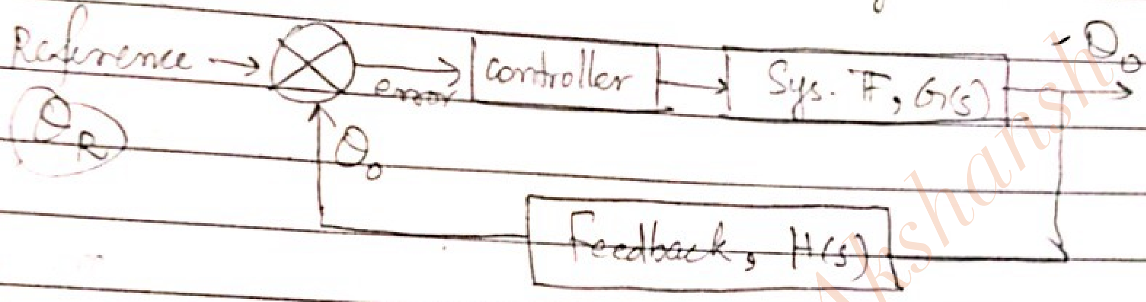






Intentional Non linearity : Relay sys :

a power amplifier, used in control sys.



(See : fig 15.10)

We see error value $\theta_R - \theta_0$ into difference amplifier (comparator) & motion of motor happens as per value of θ_R

↓

Polarity is changed to get opposite dir^n of rot^{ne}.

θ or θ

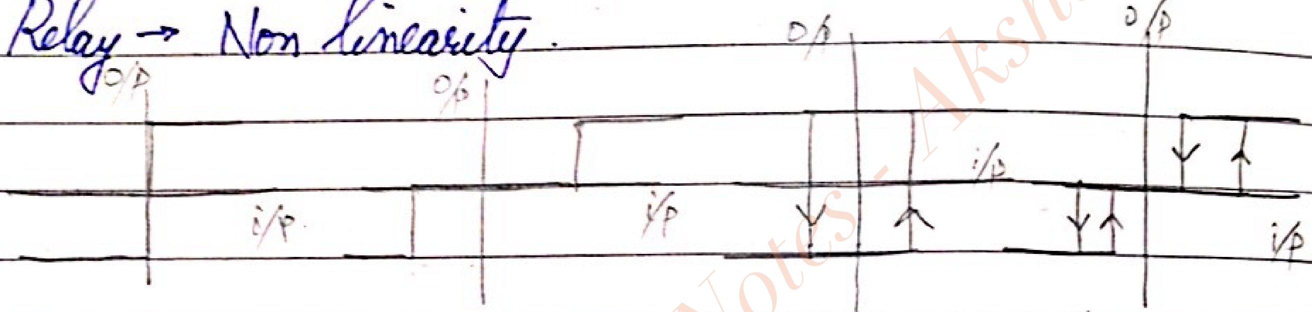
Relay $\xrightarrow{+ve\ voltage}$ Motor gets +ve voltage →

Relay $\xrightarrow{-ve\ voltage}$ Motor gets −ve voltage →

opp dir^n

* here, we are purposely introducing non linearity with help of relay

Relay → Non linearity.



Ideal relay

Relay with deadzone

Relay with hysteresis

Relay with deadzone & hysteresis